

Local-global principles for torsors over arithmetic curves

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Abstract

We consider local-global principles for torsors under linear algebraic groups, over function fields of curves over complete discretely valued fields. The obstruction to such a principle is a version of the Tate-Shafarevich group; and we show that it is finite in important cases. Moreover we obtain necessary and sufficient conditions for local-global principles to hold. The proofs use techniques from patching. We also give new applications to quadratic forms and central simple algebras.

1 Introduction

Classical local-global principles concern objects defined over number fields. As an example, let q be a quadratic form over a number field k . The famous Hasse-Minkowski theorem states that q is isotropic over k if and only if it is isotropic over each completion k_v , where v runs over all places of k . Similarly, a theorem due to Albert, Brauer, Hasse, and Noether asserts that two central simple k -algebras are isomorphic over k if and only if they become isomorphic over each k_v . There are also versions of these results for other global fields, viz. for function fields of curves defined over finite fields.

Many local-global assertions can be rephrased in terms of Galois cohomology. For a field k and an algebraic group G defined over k , the first Galois cohomology set is defined as $H^1(k, G) := H^1(\text{Gal}(k^{\text{sep}}/k), G(k^{\text{sep}}))$; here k^{sep} denotes a separable algebraic closure of k . In general, this cohomology set classifies principal homogeneous spaces (torsors) for G over k . One defines

$$\text{III}(k, G) := \ker\left(H^1(k, G) \rightarrow \prod_v H^1(k_v, G)\right),$$

where the *kernel* of a map of pointed sets is defined as the preimage of the distinguished element. Here v ranges over the set of places of k , which in the function field case consists of the discrete valuations on k . The above kernel is in general a pointed set, but it is a group (known as the *Tate-Shafarevich group*) if G is commutative. The vanishing of $\text{III}(k, G)$ is equivalent to a local-global principle for the triviality of G -torsors over k .

Concerning our two examples above, if q is a regular (i.e. non-degenerate) quadratic form of dimension n over a field k of characteristic unequal to two, then $H^1(k, \text{O}(q))$ classifies

isometry classes of all regular quadratic forms over k of the same dimension n . Moreover if the given form q is hyperbolic, then the distinguished element of $H^1(k, \mathrm{O}(q))$ corresponds to the class of hyperbolic forms. The vanishing of $\mathrm{III}(k, \mathrm{O}(q))$ is then equivalent to a local-global principle for the hyperbolicity of an n -dimensional quadratic form; and this vanishing is a consequence of Hasse-Minkowski if k is a number field. Similarly, $H^1(k, \mathrm{PGL}_n)$ classifies isomorphism classes of central simple algebras of degree n over k , with the distinguished element corresponding to the split class. The vanishing of $\mathrm{III}(k, \mathrm{PGL}_n)$ says that a central simple k -algebra of degree n is split if it is split over each k_v ; and this is a consequence of Albert-Brauer-Hasse-Noether if k is a number field. If G is any linear algebraic group over a global field k , then $\mathrm{III}(k, G)$ need not be trivial, but is finite ([BS64] in the number field case; [Oes84], [BP90], [Con10] in the function field case). As another example, for an abelian variety A over a number field k , the group $\mathrm{III}(k, A)$ is conjectured to be finite, with its order being related to the behavior of an associated L -function.

In this paper, we consider local-global principles for torsors under linear algebraic groups G that are defined over one-variable function fields F over a complete discretely valued field K with valuation ring T (whose residue field is not necessarily finite). That is, we study

$$\mathrm{III}(F, G) := \ker\left(H^1(F, G) \rightarrow \prod_v H^1(F_v, G)\right),$$

which was considered in [COP02] and [CGP04]. In particular, we ask whether $\mathrm{III}(F, G)$ is finite and when it is trivial. We do so by relating $\mathrm{III}(F, G)$ to

$$\mathrm{III}_0(\widehat{X}, G) := \ker\left(H^1(F, G) \rightarrow \prod_{P \in X} H^1(F_P, G)\right).$$

Here X is the closed fiber of a regular projective T -curve \widehat{X} having function field F , and F_P denotes the fraction field of the complete local ring of \widehat{X} at a point $P \in X$. We obtain the strongest results in the case when the group G is rational (i.e., each connected component is a rational variety; see the beginning of Section 4). In that case, we show that $\mathrm{III}_0(\widehat{X}, G)$ is independent of the regular model \widehat{X} (Corollary 7.8 and its comment). We also show that $\mathrm{III}_0(\widehat{X}, G)$ is finite for such groups G , and we give a necessary and sufficient condition for it to vanish (Corollary 6.5). In fact, $\mathrm{III}_0(\widehat{X}, G)$ is contained in $\mathrm{III}(F, G)$ (Proposition 8.2), and so $\mathrm{III}_0(\widehat{X}, G)$ must vanish if $\mathrm{III}(F, G)$ does. In special cases we show that $\mathrm{III}(F, G) = \mathrm{III}_0(\widehat{X}, G)$; and if G is rational we then have that $\mathrm{III}(F, G)$ is finite and we obtain a necessary and sufficient condition for a local-global principle with respect to valuations to hold (Theorems 8.7 and 8.10). New applications are given, in particular local-global principles for quadratic forms and central simple algebras. These were motivated by [CPS08], where results from [HHK09] were used to obtain local-global results for quadratic forms in terms of discrete valuations.

Our approach in studying $\mathrm{III}_0(\widehat{X}, G)$ is to use the method of patching over fields that was developed in [HH10], and which was afterwards extended in [HHK09] for the purpose of obtaining local-global principles and related results about quadratic forms and central

simple algebras. The local-global principles in [HHK09] were stated for a *finite* collection of overfields of F , and were thus somewhat different from those encoded in $\text{III}(F, G)$ and $\text{III}_0(\widehat{X}, G)$.

In order to study such local-global principles for torsors, we extend the patching machinery to torsors under linear algebraic groups. More precisely we show that patching for torsors will hold in any setup in which patching of vector spaces does (Theorem 2.3). This appears to be the first patching theorem for objects that are not finite (unlike e.g. GAGA and Grothendieck's Existence Theorem), and should be of independent interest. We show that in situations where patching holds, a local-global principle for torsors automatically yields one for more general homogeneous spaces, and so the results presented here generalize those of [HHK09].

In [HHK09], for an arbitrary K -curve, a local-global principle was given only for *connected* rational linear algebraic groups, whereas for special curves like the line, no connectedness assumption on the group was necessary. Moreover, an example due to Colliot-Thélène (see [CPS08, Remark 4.4], [HHK09, Example 4.4]) showed that local-global principles under disconnected rational groups may fail if F has extensions with a certain splitness property. The results of this paper explain the precise hypotheses that can be made either on the curve or on the group to ensure a local-global principle, as well as describing the kernel of the local-global map in the general case.

Structure of the manuscript. The paper is organized as follows. Section 2 extends the machinery of patching to the situation of torsors under linear algebraic groups, and uses this to obtain a six-term exact sequence on the level of H^0 and H^1 . This is done in an abstract setup, for certain inverse systems of fields. The section also relates the corresponding local-global maps for a linear algebraic group G to that of quotient groups. Section 3 applies the abstract results to the patching setup used in [HHK09], over the function field of an arithmetic curve, and introduces a patching analog $\text{III}_{\mathcal{P}}(\widehat{X}, G)$ of $\text{III}_0(\widehat{X}, G)$. In the case of rational linear algebraic groups, the study of $\text{III}_{\mathcal{P}}(\widehat{X}, G)$ can be reduced to the case of finite groups; this is done in Section 4. Section 5 introduces $\text{III}_0(\widehat{X}, G)$ and shows that it agrees with $\text{III}_{\mathcal{P}}(\widehat{X}, G)$ for rational groups. It then gives a description of $\text{III}_0(\widehat{X}, G)$ in terms of a certain quotient $\pi_1^{\text{split}}(\widehat{X})$ of the étale fundamental group $\pi_1(\widehat{X})$. In Section 6, the reduction graph Γ of a model \widehat{X} is used to show that $\text{III}_0(\widehat{X}, G)$ is finite and can be described explicitly for rational groups; and this description shows that $\text{III}_0(\widehat{X}, G)$ is trivial if and only if either G is connected or Γ is a tree. Then in Section 7, it is shown for G rational that $\text{III}_0(\widehat{X}, G)$ is independent of the choice of a regular model \widehat{X} of F , as a consequence of proving that $\text{III}_0(\widehat{X}, G) = H^1(F^{\text{split}}/F, G)$, where F^{split} is the maximal algebraic field extension of F that is completely split over every discrete valuation. Having described the structure of $\text{III}_0(\widehat{X}, G)$, we analyze the relationship between $\text{III}(F, G)$ and $\text{III}_0(\widehat{X}, G)$ in Section 8, showing in several situations that they are equal. In Section 9, we turn to homogeneous spaces which are not (necessarily) torsors. We also apply our results to the case of quadratic forms, which is where Colliot-Thélène's original example occurred, and we prove local-global results concerning the Witt group and the Witt index. Finally, we give applications that

assert that central simple algebras are isomorphic if and only if they are locally isomorphic.

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2 Patching Problems and Torsors

In this section we consider patching problems for torsors under linear algebraic groups, where we use the term *linear algebraic group* to denote a smooth affine group scheme of finite type (cf. [Bor91]); or equivalently, a geometrically reduced closed subgroup of some general linear group. We show that these patching problems can be solved and that there is an equivalence of categories (Theorem 2.3). Since the coordinate ring of a torsor is in general infinite dimensional as a vector space, this result goes beyond the type of patching theorem obtained in [HH10, Section 7]. Using these results, we obtain a six term exact sequence at the level of H^0 and H^1 (Theorem 2.4).

Patching considers a field F and a collection of overfields. It asserts that if compatible structures are given over these overfields, then there is a structure over F inducing all of them compatibly. We first recall how it can be described in the situation in which vector spaces are the structure under consideration. Let $\mathcal{F} := \{F_i\}_{i \in I}$ be a finite system of fields and inclusions, whose inverse limit is a field F .

For $i, j \in I$, write $i \succ j$ if we are given a proper inclusion $F_i \hookrightarrow F_j$. A (vector space) *patching problem* for \mathcal{F} is a system $\mathcal{V} := \{V_i\}_{i \in I}$ of finite dimensional F_i -vector spaces for $i \in I$, together with F_j -isomorphisms $\nu_{i,j} : V_i \otimes_{F_i} F_j \rightarrow V_j$ for all $i \succ j$ in I . (This is equivalent to the definition in [HH10, Section 2].) A morphism of patching problems is defined in the obvious way. We write $\text{PP}(\mathcal{F})$ for the category of vector space patching problems for \mathcal{F} .

In the above setup, there is a functor

$$\beta : \text{Vect}(F) \rightarrow \text{PP}(\mathcal{F})$$

from the category of finite dimensional F -vector spaces to the category of vector space patching problems for \mathcal{F} defined by base change. A *solution* to a patching problem \mathcal{V} for \mathcal{F} is an F -vector space V such that $\beta(V)$ is isomorphic to \mathcal{V} in the category of patching problems. (See [HH10, Section 2] for more details.) If β is an equivalence of categories, then every patching problem has a unique solution up to isomorphism.

The simplest type of patching problem consists of two fields F_1, F_2 with a common overfield F_0 and intersection F . In this situation, β is an equivalence of categories if and only if every matrix $A_0 \in \text{GL}_n(F_0)$ can be factored as $A_1^{-1}A_2$ with $A_i \in \text{GL}_n(F_i)$. (See [HH10, Section 2].)

In this paper, we are concerned with a special type of inverse system.

Definition 2.1. We say that a finite inverse system of fields is a *factorization inverse system* if its index set I has the following property: There is a partition $I = I_v \cup I_e$ into a disjoint union such that for each index $k \in I_e$, there are exactly two elements $i, j \in I_v$ for which $i, j \succ k$, and there are no other relations in I .

A factorization inverse system determines a (multi-)graph whose vertices are the elements of I_v and whose edges are the elements of I_e , with the vertices of an edge $k \in I_e$ corresponding to the elements $i, j \in I_v$ such that $i, j \succ k$. Conversely, a graph determines a factorization inverse system.

Notice that the simple inverse system consisting of two fields F_1, F_2 with a common overfield F_0 is a factorization inverse system. In fact, our interest in factorization inverse systems is based on a generalization of the factorization property for $\mathrm{GL}_n(F_0)$ mentioned above. Namely, if $\{F_i\}_{i \in I}$ is a factorization inverse system, then the base change functor β is an equivalence of categories if and only if the *simultaneous factorization* property holds, as we explain next.

Fix once and for all for each index $k \in I_e$ a labeling $l = l_k, r = r_k$ of the two elements of I_v that dominate k in the inverse system (this corresponds to orienting the edges of the graph, i.e., assigning to each edge a left and right vertex), thereby associating to each $k \in I_e$ an ordered triple (l, r, k) . The oriented system is then determined by the set S_I of such triples.

Note that giving a patching problem for \mathcal{F} is then equivalent to giving a collection of F_i -vector spaces V_i for $i \in I_v$ together with isomorphisms $\mu_k : V_l \otimes_{F_l} F_k \rightarrow V_r \otimes_{F_r} F_k$, where (l, r, k) ranges over the set S_I . In the notation above, $\mu_k = \nu_{r,k}^{-1} \nu_{l,k}$. (Compare the discussion at the end of Section 2 of [HH10].)

In the above situation, we say that *simultaneous factorization* holds for \mathcal{F} if for any $n \geq 1$ and any collection of matrices $A_k \in \mathrm{GL}_n(F_k)$ (for $k \in I_e$), there exist matrices $A_i \in \mathrm{GL}_n(F_i)$ for all $i \in I_v$ such that $A_k = A_r^{-1} A_l \in \mathrm{GL}_n(F_k)$ for all $(l, r, k) \in S_I$, with respect to the inclusions $F_l, F_r \hookrightarrow F_k$. Notice that an index $i \in I_v$ may dominate several indices $k \in I_e$, but the matrix A_i is the same for all corresponding inclusions (hence the use of the term *simultaneous*).

Proposition 2.2. *The functor $\beta : \mathrm{Vect}(F) \rightarrow \mathrm{PP}(\mathcal{F})$ is an equivalence of categories if and only if simultaneous factorization holds for \mathcal{F} .*

Proof. The case of the simple factorization system $F_1, F_2 \subset F_0$ was shown in Proposition 2.1 of [Ha84] (see also Proposition 2.1 of [HH10]). We now discuss the general case, which is similar. First suppose that β is an equivalence of categories, and consider a collection of matrices $A_k \in \mathrm{GL}_n(F_k)$ for $k \in I_e$. For each $i \in I_v$ choose an n -dimensional F_i -vector space V_i with basis B_i . For each triple $(l, r, k) \in S_I$, and with respect to the bases B_l, B_r , the matrix A_k defines a vector space isomorphism $\mu_k : V_l \otimes_{F_l} F_k \rightarrow V_r \otimes_{F_r} F_k$. These isomorphisms define a patching problem for \mathcal{F} , which then has a solution V over F . We thus have isomorphisms $\alpha_i : V \otimes_F F_i \rightarrow V_i$ that are compatible with the maps μ_k . Choose a basis B of V , and let $A_i \in \mathrm{GL}_n(F_i)$ be the matrix corresponding to α_i with respect to B, B_i . Then $A_k = A_r^{-1} A_l$ for each $(l, r, k) \in S_I$, proving simultaneous factorization.

Conversely, assume that simultaneous factorization holds, and consider a patching problem $\mathcal{V} := \{V_i\}_{i \in I}$ for \mathcal{F} , corresponding to isomorphisms μ_k as above. Choose bases B_i for each V_i , and let A_k be the transition matrix between V_r and V_l for each triple $(l, r, k) \in S_I$. By hypothesis, there exist matrices $A_i \in \mathrm{GL}_n(F_i)$ for all $i \in I_v$ such that $A_k = A_r^{-1}A_l$ for each $(l, r, k) \in S_I$. Adjusting the bases B_i by the matrices A_i , we obtain new bases B'_i for V_i such that B'_l, B'_r have the same images in V_k for each $(l, r, k) \in S_I$. This yields a solution to the patching problem, given by an n -dimensional F -vector space V with a basis B , together with the isomorphisms $V \otimes_F F_i \rightarrow V_i$ that take B to B'_i . Thus every patching problem has a solution, and so β is surjective on isomorphism classes of objects. It is now straightforward to check that β is bijective on morphisms, as in the proof of Proposition 2.1 of [Ha84]. \square

We next consider a different category of patching problems. If G is a linear algebraic group over F and L/F is a field extension, we write G_L for the base change of G to L . We define a *G -torsor patching problem* for \mathcal{F} to consist of a system of G_{F_i} -torsors \mathcal{T}_i together with F_j -isomorphisms of G_{F_j} -torsors $\nu_{i,j} : \mathcal{T}_i \times_{F_i} F_j \rightarrow \mathcal{T}_j$ for all pairs $i \succ j$ (i.e. such that $F_i \subset F_j$). A G -torsor \mathcal{T} over F induces a G -torsor patching problem by base change; and a *solution* to a given G -torsor patching problem consists of a G -torsor \mathcal{T} that induces the given patching problem up to isomorphism. Again, the G -torsor patching problems form a category with the obvious definition of a morphism.

In our situation, the inverse system consists of pairs of inclusions $F_l \subset F_k$ and $F_r \subset F_k$, for $(l, r, k) \in S_I$. Giving a G -torsor patching problem is therefore equivalent to giving a G_{F_i} -torsor \mathcal{T}_i for each $i \in I$ together with G_{F_k} -torsor isomorphisms $\mu_k : \mathcal{T}_l \times_{F_l} F_k \rightarrow \mathcal{T}_r \times_{F_r} F_k$ for each $(l, r, k) \in S_I$. Again, $\mu_k = \nu_{r,k}^{-1} \nu_{l,k}$.

Recall that G -torsors over a field are classified by its first Galois cohomology set. Regard G as a closed subgroup of GL_n , and consider an extension field E/F . Since $H^1(E, \mathrm{GL}_n)$ is trivial by Hilbert's Theorem 90 ([KMRT98, Theorem 29.2]), Corollaire 1 of [Ser73, I.5.4, Proposition 36] yields a bijection of pointed sets

$$\mathrm{GL}_n(E) \backslash H^0(E, \mathrm{GL}_n/G) \xrightarrow{\sim} H^1(E, G).$$

Thus G -torsors are parametrized by $\mathrm{Gal}(E^{\mathrm{sep}}/E)$ -invariant translates hG of G , with $h \in \mathrm{GL}_n(E^{\mathrm{sep}})$.

Recall also that the coset space GL_n/G carries the structure of a quasi-projective variety together with a quotient morphism $\pi : \mathrm{GL}_n \rightarrow \mathrm{GL}_n/G$. For every field extension E/F , and every $h \in \mathrm{GL}_n(E^{\mathrm{sep}})$, the fiber of π over $\pi(h)$ is the translate $hG \subseteq \mathrm{GL}_n$; and this fiber is $\mathrm{Gal}(E^{\mathrm{sep}}/E)$ -invariant if and only if $\pi(h)$ is. Equivalently, hG defines an element of $H^0(E, \mathrm{GL}_n/G)$ if and only if $\pi(h)$ is defined over E . For details see [Bor91], Theorem II.6.8 and its proof.

Using the above, we now prove that patching for vector spaces implies patching for torsors.

Theorem 2.3. *Let $\mathcal{F} := \{F_i\}_{i \in I}$ be a factorization inverse system with inverse limit F , and let G be a linear algebraic group over F . If the base change functor $\beta : \mathrm{Vect}(F) \rightarrow \mathrm{PP}(\mathcal{F})$ is an equivalence of categories, then so is the functor from the category of G -torsors over F*

to the category of G -torsor patching problems for \mathcal{F} . In particular, every G -torsor patching problem has a solution that is unique up to isomorphism. Moreover, on the level of coordinate rings, this solution is given by taking the inverse limit.

Proof. View G as a closed subgroup of $\mathrm{GL}_{n,F}$. The main effort is to show that every G -torsor patching problem has a solution. By the above discussion, we may assume that the given patching problem consists of G -torsors $\mathcal{T}_i = h_i G_{F_i}$ which are $\mathrm{Gal}(F_i^{\mathrm{sep}}/F_i)$ -invariant orbits of elements $h_i \in \mathrm{GL}_n(F_i^{\mathrm{sep}})$, for $i \in I_v$, together with isomorphisms $\mu_k : \mathcal{T}_l \times_{F_l} F_k \rightarrow \mathcal{T}_r \times_{F_r} F_k$ for each triple $(l, r, k) \in S_I$.

For each $(l, r, k) \in S_I$, the fields F_l and F_r include into F_k ; thus the elements h_l, h_r may be viewed as elements of $\mathrm{GL}_n(F_k^{\mathrm{sep}})$. Consider the collection of elements $g_k := \mu_k(h_l)h_l^{-1} \in \mathrm{GL}_n(F_k^{\mathrm{sep}})$. Left multiplication by g_k determines a morphism of torsors $\lambda_k : h_l G_{F_k} \rightarrow g_k h_l G_{F_k} = h_r G_{F_k}$ (a priori defined over F_k^{sep}) which sends $h_l \in \mathcal{T}_l(F_k^{\mathrm{sep}})$ to $\mu_k(h_l) \in \mathcal{T}_r(F_k^{\mathrm{sep}})$ by definition of g_k . Since a morphism of torsors is determined by the image of one point, $\lambda_k = \mu_k$. In particular, λ_k is also defined over F_k (i.e. it commutes with the action of $\mathrm{Gal}(F_k^{\mathrm{sep}}/F_k)$), and hence $g_k \in \mathrm{GL}_n(F_k)$.

By the hypothesis on vector space patching problems for \mathcal{F} , Proposition 2.2 implies that there is a collection of elements $g_i \in \mathrm{GL}_n(F_i)$ for $i \in I_v$ such that $g_k = g_r^{-1}g_l \in \mathrm{GL}_n(F_k)$ for each triple $(l, r, k) \in S_I$. For $i \in I_v$, let $h'_i = g_i h_i$ and let $\lambda_i : h_i G_{F_i} \rightarrow h'_i G_{F_i}$ be left multiplication by g_i . Since $g_k = g_r^{-1}g_l$, the morphisms $(\lambda_r)_{F_k}^{-1}$ and $\mu_k \circ (\lambda_l)_{F_k}^{-1} : h'_l G_{F_k} = h'_r G_{F_k} \rightarrow h_r G_{F_k}$ are the same.

For $k \in I_e$, write h'_k for the image of h'_r in $\mathrm{GL}_n(F_k^{\mathrm{sep}})$. The translates $h'_i G_{F_i}$ are invariant under $\mathrm{Gal}(F_i^{\mathrm{sep}}/F_i)$, for $i \in I$. Consider their images Q_i in the F -variety GL_n/G . Thus Q_i is a $\mathrm{Gal}(F_i^{\mathrm{sep}}/F_i)$ -invariant point in $(\mathrm{GL}_n/G)(F_i^{\mathrm{sep}})$, and so it is defined over F_i . That is, the points Q_i , for $i \in I$, form a compatible system of F_i -points of the F -variety GL_n/G . Since there are finitely many such points, they lie in an affine open subset of the quasi-projective variety GL_n/G , which can be viewed as contained in \mathbb{A}_F^m for some m . For each $s = 1, \dots, m$, the s -th coordinates of these points form an inverse system of elements of the fields F_i . But by assumption, the inverse limit of the system \mathcal{F} is F . Hence the points Q_i define an F -point $Q \in (\mathrm{GL}_n/G)(F)$. The fiber over this F -point is a $\mathrm{Gal}(F^{\mathrm{sep}}/F)$ -invariant translate hG of G for some $h \in \mathrm{GL}_n(F^{\mathrm{sep}})$, such that $(hG)_{F_i} = h'_i G_{F_i}$ for each $i \in I$.

Consider the G -torsor $\mathcal{T} = hG$ defined over F . Since $(\lambda_r)_{F_k}^{-1} = \mu_k \circ (\lambda_l)_{F_k}^{-1}$ for each $(l, r, k) \in S_I$, the torsor \mathcal{T} , together with the maps $\lambda_i^{-1} : (hG)_{F_i} \rightarrow h_i G_{F_i}$, defines a solution to the given patching problem.

We have shown that the base change functor defines a surjection on isomorphism classes of objects. Concerning the inverse, let $\mathcal{T} = \mathrm{Spec}(A)$ be a G -torsor over F , inducing $\mathcal{T}_i = \mathrm{Spec}(A_i)$ over F_i for $i \in I$. Consider the short exact sequence $0 \rightarrow F \rightarrow \prod_{i \in I_v} F_i \rightarrow \prod_{k \in I_e} F_k$, where the F_k -entry of the image of (a_i) is $a_l - a_r$, if $(l, r, k) \in S_I$ is the triple containing k . Tensoring with A shows that $0 \rightarrow A \rightarrow \prod_{i \in I_v} A_i \rightarrow \prod_{k \in I_e} A_k$ is an exact sequence of F -vector spaces. So A is the inverse limit of the rings A_i , and therefore a solution is unique up to isomorphism. It remains to show that the functor defines a bijection on morphisms.

Given two G -torsors $\mathcal{T}, \mathcal{T}'$ over F , write $\mathcal{T} = \mathrm{Spec}(A)$ and $\mathcal{T}' = \mathrm{Spec}(A')$, and similarly for $\mathcal{T}_i, \mathcal{T}'_i$. A morphism $\mathcal{T} \rightarrow \mathcal{T}'$ corresponds to a homomorphism $A' \rightarrow A$, and similarly for

$\mathcal{T}_i \rightarrow \mathcal{T}'_i$. Since $A \subseteq A_i$, the map on morphisms is injective. Since A is the inverse limit of the A_i , the map is surjective. \square

Since F is the inverse limit of \mathcal{F} , there is an exact sequence (equalizer diagram)

$$0 \rightarrow F \rightarrow \prod_{i \in I_v} F_i \rightrightarrows \prod_{k \in I_e} F_k,$$

the double arrows corresponding to the inclusions $F_l, F_r \hookrightarrow F_k$. This yields an exact sequence

$$1 \rightarrow G(F) \rightarrow \prod_{i \in I_v} G(F_i) \rightrightarrows \prod_{k \in I_e} G(F_k),$$

which may be rewritten as an exact sequence of pointed sets

$$1 \rightarrow G(F) \rightarrow \prod_{i \in I_v} G(F_i) \rightarrow \prod_{k \in I_e} G(F_k),$$

where the F_k -component of an element in the image of $(g_i) \in \prod_{i \in I_v} G(F_i)$ is given by $g_r^{-1}g_l$, if $(l, r, k) \in S_I$. Equivalently,

$$1 \rightarrow H^0(F, G) \rightarrow \prod_{i \in I_v} H^0(F_i, G) \rightarrow \prod_{k \in I_e} H^0(F_k, G).$$

On the other hand, Theorem 2.3 yields an exact sequence

$$H^1(F, G) \rightarrow \prod_{i \in I_v} H^1(F_i, G) \rightrightarrows \prod_{k \in I_e} H^1(F_k, G).$$

The next theorem shows that these can indeed be combined to a six-term exact sequence.

Theorem 2.4. *Let \mathcal{F} be a factorization inverse system of fields with inverse limit F , such that the base change functor $\beta : \text{Vect}(F) \rightarrow \text{PP}(\mathcal{F})$ is an equivalence of categories. Then for any linear algebraic group G over F , we have an exact sequence of pointed sets*

$$\begin{array}{ccccccc} 1 & \longrightarrow & H^0(F, G) & \longrightarrow & \prod_{i \in I_v} H^0(F_i, G) & \longrightarrow & \prod_{k \in I_e} H^0(F_k, G) \\ & & & & \searrow & & \\ & & & & H^1(F, G) & \longrightarrow & \prod_{i \in I_v} H^1(F_i, G) \rightrightarrows \prod_{k \in I_e} H^1(F_k, G). \end{array}$$

Moreover, two elements $(g_k), (\tilde{g}_k) \in \prod_{k \in I_e} H^0(F_k, G)$ have the same image under the coboundary map if and only if there are $(g_i) \in \prod_{i \in I_v} G(F_i)$ such that $g_k = g_r^{-1}\tilde{g}_k g_l$ for all $(l, r, k) \in S_I$.

Proof. As above, we consider G as a closed subgroup of GL_n for some n .

First, we define a coboundary map δ . So consider an element $(g_k) \in \prod_{k \in I_e} H^0(F_k, G) = \prod_{k \in I_e} G(F_k)$. Take trivial G_{F_i} -torsors \mathcal{T}_i for each $i \in I_v$, together with points $x_i \in \mathcal{T}_i(F_i)$.

Note that the pair (\mathcal{T}_i, x_i) is unique up to a unique isomorphism. The elements g_k determine a patching problem, where for each triple $(l, r, k) \in S_I$ the isomorphism $\mu_k : \mathcal{T}_l \times_{F_l} F_k \rightarrow \mathcal{T}_r \times_{F_r} F_k$ is the one that takes x_l to $x_r g_k$. By Theorem 2.3, there is, up to isomorphism, a unique G -torsor \mathcal{T} over F that is a solution to this patching problem. The map δ is then defined to take $(g_k) \in \prod_{k \in I_e} G(F_k)$ to the class of \mathcal{T} in $H^1(F, G)$.

Next, we prove exactness at $H^1(F, G)$. The image of δ is contained in the kernel of $H^1(F, G) \rightarrow \prod H^1(F_i, G)$ since the G_{F_i} -torsor $\mathcal{T} \times_F F_i \cong \mathcal{T}_i$ is trivial in the previous paragraph. For the other direction, consider an element χ lying in the kernel of $H^1(F, G) \rightarrow \prod_{i \in I_v} H^1(F_i, G)$. This is represented by a G -torsor \mathcal{T} for which $\mathcal{T}_i := \mathcal{T} \times_F F_i$ is a trivial G_{F_i} -torsor for all $i \in I_v$. Hence we may pick points $x_i \in \mathcal{T}_i(F_i)$ for all $i \in I_v$. For every $(l, r, k) \in S_I$, there is a unique element $g_k \in G(F_k)$ for which $x_l = x_r g_k$. The patching problem determined by (g_k) , as in the previous paragraph, is given by the isomorphisms $\mu_k : \mathcal{T}_l \times_{F_l} F_k \rightarrow \mathcal{T}_r \times_{F_r} F_k$ that are induced by the identity map on \mathcal{T} . Since \mathcal{T} is a solution to this problem, the element $(g_k) \in \prod_{k \in I_e} G(F_k)$ maps to χ under δ .

For the assertion on the fibers of δ , which includes exactness at $\prod_{k \in I_e} H^0(F_k, G) = \prod_{k \in I_e} G(F_k)$, consider two elements $(g_k), (\tilde{g}_k) \in \prod_{k \in I_e} G(F_k)$. For $i \in I_v$ let $\mathcal{T}_i, x_i, \mu_i$ be as in the definition of δ for (g_k) , and similarly consider $\tilde{\mathcal{T}}_i, \tilde{x}_i, \tilde{\mu}_i$ for (\tilde{g}_k) . By Theorem 2.3, $(g_k), (\tilde{g}_k)$ map to the same element of $H^1(F, G)$ under δ if and only if the respective patching problems $(\{\mathcal{T}_i\}, \{\mu_k\}), (\{\tilde{\mathcal{T}}_i\}, \{\tilde{\mu}_k\})$ are isomorphic; i.e. if and only if there are isomorphisms $\varphi_i : \mathcal{T}_i \rightarrow \tilde{\mathcal{T}}_i$ of trivial G_{F_i} -torsors for $i \in I_v$ such that $(\varphi_r)_{F_k} \circ \mu_k = \tilde{\mu}_k \circ (\varphi_l)_{F_k}$ for each $(l, r, k) \in S_I$. Since a morphism of torsors is determined by the image of one point, giving φ_i is equivalent to giving the unique element $g_i \in G(F_i)$ such that $\varphi_i(x_i) = \tilde{x}_i g_i$. The identity on the φ_i is then equivalent to the equality $g_r g_k = \tilde{g}_k g_l$ for $(l, r, k) \in S_I$; i.e., $g_k = g_r^{-1} \tilde{g}_k g_l$. So the fibers of δ are as claimed, and the sequence is exact at $\prod_{k \in I_e} H^0(F_k, G)$. \square

In this general setup, we let

$$\text{III}_{\mathcal{F}}(G) := \ker \left(H^1(F, G) \rightarrow \prod_{i \in I_v} H^1(F_i, G) \right)$$

be the kernel of the local-global map, i.e., the set of elements that map to the trivial element. Since G need not be commutative, the sequence in the theorem is exact just as a sequence of pointed sets, not of groups; and so the kernel of a map does not determine the fibers of the map. Nevertheless, using standard techniques from nonabelian cohomology, we can describe the fibers in terms of twists.

Recall (e.g. from Section 28.C of [KMRT98] or Section I.5.3 of [Ser73]) that if G is a linear algebraic group over a field F and if $\tau \in Z^1(F, G)$ is a cocycle, then there is a naturally associated twist G^τ of G by τ . Moreover there is a bijection between $H^1(F, G)$ and $H^1(F, G^\tau)$ such that the neutral element of $H^1(F, G^\tau)$ corresponds to the class $[\tau]$ in $H^1(F, G)$; and this is functorial in F . (See [KMRT98, Proposition 28.8], or [Ser73, Proposition I.5.3.35].) Applying this discussion to F and the fields F_i , for $i \in I_v$, we obtain:

Corollary 2.5. *If $\tau \in Z^1(F, G)$, then the fiber of $H^1(F, G) \rightarrow \prod_{i \in I_v} H^1(F_i, G)$ that contains the class $[\tau] \in H^1(F, G)$ is in natural bijection with $\text{III}_{\mathcal{F}}(G^\tau)$ as pointed sets.*

Of course if τ is the trivial cocycle, then this just says that the kernel is $\text{III}_{\mathcal{F}}(G)$.

The six-term exact sequence also yields the following, which will be used to study the connection between $\text{III}_{\mathcal{F}}(G)$ and $\text{III}_{\mathcal{F}}(G/G^0)$:

Corollary 2.6. *Under the hypotheses of Theorem 2.4, consider a short exact sequence of linear algebraic groups $1 \rightarrow N \rightarrow G \rightarrow \bar{G} \rightarrow 1$ for which the map $G(L) \rightarrow \bar{G}(L)$ is surjective for every field extension L/F . Then the cohomology sequence associated to $1 \rightarrow N \rightarrow G \rightarrow \bar{G} \rightarrow 1$ induces a short exact sequence of pointed sets*

$$1 \rightarrow \text{III}_{\mathcal{F}}(N) \rightarrow \text{III}_{\mathcal{F}}(G) \rightarrow \text{III}_{\mathcal{F}}(\bar{G}) \rightarrow 1.$$

Proof. Consider the commutative diagram below, whose rows are exact by the cohomology sequence referred to in the assertion ([Ser73], I.5.5, Proposition 38), together with the surjectivity of $G \rightarrow \bar{G}$ on rational points. The columns are exact by Theorem 2.4.

$$\begin{array}{ccccccc} 1 & \longrightarrow & \prod_{k \in I_e} H^0(F_k, N) & \xrightarrow{\phi_0''} & \prod_{k \in I_e} H^0(F_k, G) & \xrightarrow{\psi_0''} & \prod_{k \in I_e} H^0(F_k, \bar{G}) \longrightarrow 1 \\ & & \downarrow \delta' & & \downarrow \delta & & \downarrow \bar{\delta} \\ 1 & \longrightarrow & H^1(F, N) & \xrightarrow{\phi_1} & H^1(F, G) & \xrightarrow{\psi_1} & H^1(F, \bar{G}) \\ & & \downarrow \iota' & & \downarrow \iota & & \downarrow \bar{\iota} \\ 1 & \longrightarrow & \prod_{i \in I_v} H^1(F_i, N) & \xrightarrow{\phi_1'} & \prod_{i \in I_v} H^1(F_i, G) & \xrightarrow{\psi_1'} & \prod_{i \in I_v} H^1(F_i, \bar{G}) \end{array}$$

The lower two rows show that the maps ϕ_1 and ψ_1 restrict to maps $\text{III}_{\mathcal{F}}(N) \rightarrow \text{III}_{\mathcal{F}}(G)$ and $\text{III}_{\mathcal{F}}(G) \rightarrow \text{III}_{\mathcal{F}}(\bar{G})$, with the former having trivial kernel and the composition of these two restrictions being trivial.

Next, we check surjectivity of $\text{III}_{\mathcal{F}}(G) \rightarrow \text{III}_{\mathcal{F}}(\bar{G})$. An element of $\text{III}_{\mathcal{F}}(\bar{G})$ is an element $\bar{\sigma} \in H^1(F, \bar{G})$ that lies in the kernel of $\bar{\iota}$. By exactness of the right hand column, $\bar{\sigma} = \bar{\delta}(\bar{g})$ for some $\bar{g} \in \prod_k H^0(F_k, \bar{G})$. Since ψ_0'' is surjective, there is some $g \in \prod_k H^0(F_k, G)$ such that $\psi_0''(g) = \bar{g}$. Let $\sigma = \delta(g) \in H^1(F, G)$. Then $\sigma \in \text{III}_{\mathcal{F}}(G)$ since $\iota(\sigma) = \iota\delta(g)$ is trivial; and $\psi_1(\sigma) = \psi_1\delta(g) = \bar{\delta}\psi_0''(g) = \bar{\delta}(\bar{g}) = \bar{\sigma}$. So indeed the map is surjective.

Finally, for exactness at $\text{III}_{\mathcal{F}}(G)$, note that an element $\sigma \in \ker(\text{III}_{\mathcal{F}}(G) \rightarrow \text{III}_{\mathcal{F}}(\bar{G}))$ is an element of $H^1(F, G)$ lying in the kernels of ι and ψ_1 . Since the middle row is exact, $\sigma = \phi_1(\tau)$ for some $\tau \in H^1(F, N)$. Now $\iota'(\tau)$ is trivial since $\phi_1'\iota'(\tau) = \iota\phi_1(\tau) = \iota(\sigma)$ is trivial and since ϕ_1' has trivial kernel. So τ is in $\text{III}_{\mathcal{F}}(N)$, and σ is in the image of $\text{III}_{\mathcal{F}}(N)$. \square

We will be interested in cases when the map $\text{III}_{\mathcal{F}}(G) \rightarrow \text{III}_{\mathcal{F}}(\bar{G})$ in Corollary 2.6 is a bijection. The next proposition gives a criterion for this.

Proposition 2.7. *Under the hypotheses of Corollary 2.6, the following are equivalent:*

- (i) *The map $\text{III}_{\mathcal{F}}(G) \rightarrow \text{III}_{\mathcal{F}}(\bar{G})$ is a bijection of pointed sets.*
- (ii) *For every pair of elements $(g_k), (\tilde{g}_k) \in \prod_{k \in I_e} G(F_k)$ that have the same image in $\prod_{k \in I_e} \bar{G}(F_k)$, there exist elements $(g_i) \in \prod_{i \in I_v} G(F_i)$ such that $g_k = g_r^{-1} \tilde{g}_k g_l$ whenever $(l, r, k) \in S_I$.*

Proof. Assume that $\text{III}_{\mathcal{F}}(G) \rightarrow \text{III}_{\mathcal{F}}(\bar{G})$ is a bijection, and consider $(g_k), (\tilde{g}_k) \in \prod_{k \in I_e} G(F_k)$ as in (ii). By Theorem 2.4, these define elements $\alpha, \tilde{\alpha} \in \text{III}_{\mathcal{F}}(G)$, which map to the same element in $\text{III}_{\mathcal{F}}(\bar{G})$ since $(g_k), (\tilde{g}_k)$ have the same image in $\prod_{k \in I_e} \bar{G}(F_k)$. By assumption, this implies $\alpha = \tilde{\alpha}$, which in turn implies that the images of (g_k) and (\tilde{g}_k) under the coboundary map are the same. Now use the last assertion of Theorem 2.4.

On the other hand, assume (ii). By Corollary 2.6, it suffices to show injectivity. Consider two elements $\alpha, \tilde{\alpha} \in \text{III}_{\mathcal{F}}(G)$ that have the same image in $\text{III}_{\mathcal{F}}(\bar{G})$. Let $(g_k), (\tilde{g}_k) \in \prod_{\mathcal{F}} G(F_k)$ be preimages under the coboundary map δ in Theorem 2.4, and let $(\bar{g}_k), (\tilde{\bar{g}}_k)$ be their images in $\bar{G}(F_k)$. By assumption, (\bar{g}_k) and $(\tilde{\bar{g}}_k)$ induce the same element in $\text{III}_{\mathcal{F}}(\bar{G})$; hence by the last assertion of Theorem 2.4, there exist $(\bar{h}_i) \in \bar{G}(F_i)$ for $i \in I_v$ such that $\bar{g}_k = \bar{h}_r^{-1} \tilde{\bar{g}}_k \bar{h}_l$ whenever $(l, r, k) \in S_I$. By the surjectivity hypothesis on $G \rightarrow \bar{G}$, there exist preimages $h_i \in G(F_i)$ for the elements \bar{h}_i . Replacing \tilde{g}_k by $h_r \tilde{g}_k h_l$, we may assume that (g_k) and (\tilde{g}_k) have the same image in $\prod_{k \in I_e} \bar{G}(F_k)$. By the assumption of (ii) combined with the last assertion of Theorem 2.4, this shows that (g_k) and (\tilde{g}_k) map to the same element under δ , i.e., $\alpha = \tilde{\alpha}$, thereby proving injectivity. \square

Before we apply the results of this section in a more concrete setup, we state one more corollary to Theorem 2.4.

Corollary 2.8. *Under the hypotheses of Theorem 2.4, assume that $\text{III}_{\mathcal{F}}(G) = 1$ and that H is a G -variety over F for which $G(F_k)$ acts transitively on $H(F_k)$ for all $k \in I_e$. If $H(F_i) \neq \emptyset$ for each $i \in I_v$, then $H(F) \neq \emptyset$.*

Proof. By the six-term exact sequence, the triviality of $\text{III}_{\mathcal{F}}(G)$ is equivalent to a simultaneous factorization property for the group G (rather than just GL_n). The proof is then the same as in [HHK09], Theorem 3.7 (where an additional hypothesis on the group ensured the simultaneous factorization property). \square

In other words, a local-global principle for torsors implies a local-global principle for homogeneous spaces (in our setup), hence the corollary allows us to restrict our attention to torsors. We will return to other homogeneous spaces in Section 9.1.

3 Patching over Arithmetic Curves

In the previous section, we saw that patching for vector spaces implies patching for torsors, and that this yields a six-term exact sequence for Galois cohomology. We wish to apply these results to linear algebraic groups defined over the function field F of an arithmetic curve. In order to do so, we use a setup introduced in [HH10].

Notation 3.1. Let T be a complete discrete valuation ring with uniformizer t , fraction field K , and residue field k . Let F be a one-variable function field over K , and let \widehat{X} be a *normal model* of F , i.e. a normal connected projective T -curve with function field F . For each point P of the closed fiber $X \subset \widehat{X}$, let \widehat{R}_P be the completion of the local ring of \widehat{X} at P

(with respect to its maximal ideal), and F_P be the fraction field of \widehat{R}_P . For each subset U of X that is contained in an irreducible component of X and does not meet other components, let \widehat{R}_U be the t -adic completion of the subring of F consisting of rational functions that are regular on U , and let F_U be its fraction field. For each branch of X at a closed point P , i.e., for each height one prime \wp of \widehat{R}_P that contains t , let \widehat{R}_\wp be the completion of the local ring of \widehat{R}_P at \wp , and let F_\wp be its fraction field.

In the setup of Notation 3.1, in [HH10] and [HHK09], we considered an inverse system of fields, arising from a choice as follows:

Notation 3.2. In Notation 3.1, let \mathcal{P} be a non-empty finite set of closed points of X that contains all the closed points at which distinct irreducible components of X meet. Let \mathcal{U} be the set of connected components of the complement of \mathcal{P} in X . Let \mathcal{B} be the set of branches of X at points of \mathcal{P} .

This yields a finite inverse system of fields F_P, F_U, F_\wp of F (for $P \in \mathcal{P}$, $U \in \mathcal{U}$, $\wp \in \mathcal{B}$), where $F_P, F_U \subset F_\wp$ if \wp is a branch of X at P lying on the closure of U . The system is indeed a factorization inverse system.

In [HH10], Proposition 6.3, it was shown that the inverse limit of this system is F , provided that $\mathcal{P} = f^{-1}(\infty)$ for some finite morphism $f : \widehat{X} \rightarrow \mathbb{P}_T^1$, where ∞ is the point at infinity on the closed fiber of \mathbb{P}_T^1 . But in fact, this last assumption is satisfied automatically, as the following strengthening of [HH10, Proposition 6.6] shows:

Proposition 3.3. *In the situation of Notation 3.1, let S be a finite set of closed points of \widehat{X} and write ∞ for the point at infinity on $\mathbb{P}_k^1 \subset \mathbb{P}_T^1$. Then there is a finite T -morphism $f : \widehat{X} \rightarrow \mathbb{P}_T^1$ such that $S = f^{-1}(\infty)$ if and only if S meets each irreducible component of X non-trivially. In particular, there is such an f for $S = \mathcal{P}$ as in Notation 3.2.*

Proof. By [Liu02], Corollary 5.3.8 (or by [Hts77], Exercise III.5.7(c)), a divisor on a proper scheme is ample if and only if its restriction to each irreducible component is ample. Since an effective divisor on an irreducible projective curve over a field is ample if and only if it is non-zero, it follows that an effective divisor on X is ample if and only if it meets each irreducible component of X non-trivially.

Since ∞ defines an ample divisor on \mathbb{P}_k^1 , the forward implication of the proposition now follows from the fact that the inverse image of an ample divisor under a finite surjective morphism is ample ([Liu02], Remark 5.3.9, or [Hts77], Exercise III.5.7(d)).

We now prove the reverse implication; so assume that S meets each irreducible component of X non-trivially. For each point $P \in S$, the maximal ideal \mathfrak{m}_P of the local ring $\mathcal{O}_{\widehat{X},P}$ strictly contains the ideal I_P that defines the reduced closed fiber of \widehat{X} in $\text{Spec}(\mathcal{O}_{\widehat{X},P})$; so we may choose an element $r_P \in \mathfrak{m}_P$ that does not lie in I_P . The element r_P defines an effective Cartier divisor \widehat{D}_P on $\text{Spec}(\mathcal{O}_{\widehat{X},P})$, which we may view as an effective Cartier divisor on \widehat{X} whose support meets X precisely at P . Let $\widehat{D} = \sum_{P \in S} \widehat{D}_P$. The restriction of \widehat{D} to X has support S , and so meets each irreducible component of X non-trivially. Thus this restriction is an ample divisor on X . Hence by [Liu02], Corollary 5.3.24 (or by [Gro61b], Théorème 4.7.1), \widehat{D}

is an ample divisor on \widehat{X} . Replacing \widehat{D} by a multiple (which corresponds to replacing each r_P by a power), we may assume that \widehat{D} is very ample ([Gro61a], Proposition 4.5.10(ii)(e')). Hence for some n there is an embedding $i : \widehat{X} \rightarrow \mathbb{P}_T^n$ such that $\widehat{D} = i^*(H_1)$ for some T -hyperplane $H_1 \subset \mathbb{P}_T^n$, given by a linear form F_1 over T . Choose a basic open subset of \mathbb{P}_k^n that contains the finite set $i(S)$. Its complement in \mathbb{P}_k^n is a k -hypersurface, hence is given by a homogeneous k -form of some degree d . Lifting the coefficients of this form from k to T , we obtain a T -hypersurface $H_2 \subset \mathbb{P}_T^n$, defined by a homogeneous T -form F_2 of degree d , such that the support of $i^*(H_2)$ does not meet S and hence is disjoint from the support of \widehat{D} . Consider the rational function $f := i^*(F_2/F_1^d)$ on \widehat{X} , whose zero divisor is $i^*(H_2)$ and whose pole divisor is $i^*(dH_1) = d\widehat{D}$. Since those divisors have disjoint supports, f defines a T -morphism $\widehat{X} \rightarrow \mathbb{P}_T^1$, such that the inverse image of $\infty \in \mathbb{P}_k^1$ is S . It remains to show that f is finite.

The pole locus of f on \widehat{X} , viz. the support of \widehat{D} , meets each irreducible component of X at a point of S . The zero locus of f on \widehat{X} is the support of the very ample divisor $i^*(H_2)$, which therefore also meets each irreducible component of X non-trivially. Hence f is non-constant on each irreducible component of X , having both a zero and a pole there. Similarly, f is non-constant on the general fiber of \widehat{X} , which is irreducible since \widehat{X} itself is. Thus f does not contract any irreducible component of a fiber of $\widehat{X} \rightarrow T$, and so is a quasi-finite (i.e. finite-to-one) T -morphism. But since the proper morphism $\widehat{X} \rightarrow T$ factors through $f : \widehat{X} \rightarrow \mathbb{P}_T^1$, it follows that f is proper ([Hts77], Corollary II.4.8(e)). Being quasi-finite and proper, f is finite by [Gro66], Théorème 8.11.1.

The last part of the proposition is immediate from the first part if X is irreducible, since \mathcal{P} is non-empty. On the other hand, if X is reducible, then each component must meet some other component by connectivity of X , and thus each component contains a point of \mathcal{P} . So again the hypothesis of the first part is satisfied. \square

From the observation prior to Proposition 3.3 together with [HH10], Theorem 6.4, we obtain:

Corollary 3.4. *The inverse system given by Notation 3.2 is a factorization inverse system with inverse limit F . For this system, the base change functor $\beta : \text{Vect}(F) \rightarrow \text{PP}(\mathcal{F})$ is an equivalence of categories.*

Moreover, for each $\wp \in \mathcal{B}$, exactly one of the two indices dominating it is from \mathcal{P} and one from \mathcal{U} . In particular, there is a natural and uniform way to order those indices: We consider triples (P, U, \wp) such that \wp is a branch at P lying on the closure of U .

As a consequence of Corollary 3.4, the results of Section 2 can be applied to the inverse system given in Notation 3.2. In particular, we obtain the following Mayer-Vietoris type sequence from Theorem 2.4:

Theorem 3.5. *Under Notation 3.2, and for any linear algebraic group G over F , we have*

an exact sequence of pointed sets

$$\begin{array}{c}
1 \longrightarrow H^0(F, G) \longrightarrow \prod_{P \in \mathcal{P}} H^0(F_P, G) \times \prod_{U \in \mathcal{U}} H^0(F_U, G) \longrightarrow \prod_{\wp \in \mathcal{B}} H^0(F_\wp, G) \\
\searrow \hspace{10em} \swarrow \\
H^1(F, G) \longrightarrow \prod_{P \in \mathcal{P}} H^1(F_P, G) \times \prod_{U \in \mathcal{U}} H^1(F_U, G) \rightrightarrows \prod_{\wp \in \mathcal{B}} H^1(F_\wp, G).
\end{array}$$

Moreover, two elements $(g_\wp), (\tilde{g}_\wp) \in \prod_{\wp \in \mathcal{B}} H^0(F_\wp, G)$ have the same image under the coboundary map if and only if there exist $g_\xi \in G(F_\xi)$ for each $\xi \in \mathcal{P} \cup \mathcal{U}$ such that $g_\wp = g_U^{-1} \tilde{g}_\wp g_P$ whenever \wp is a branch at P lying on the closure of U .

In this setup, the kernel of the local-global map will be denoted by

$$\text{III}_{\mathcal{P}}(\hat{X}, G) := \ker(H^1(F, G) \rightarrow \prod_{\xi \in \mathcal{P} \cup \mathcal{U}} H^1(F_\xi, G)).$$

This equals $\text{III}_{\mathcal{F}}(G)$, for the inverse system of fields \mathcal{F} associated to $\mathcal{P}, \mathcal{U}, \mathcal{B}$. Note that $\mathcal{P} \subset X$ determines \mathcal{U} , and so the notation need not make explicit mention of \mathcal{U} .

Corollary 3.6. *The coboundary of the above exact sequence induces a bijection*

$$\prod_{U \in \mathcal{U}} G(F_U) \setminus \prod_{\wp \in \mathcal{B}} G(F_\wp) / \prod_{P \in \mathcal{P}} G(F_P) \rightarrow \text{III}_{\mathcal{P}}(\hat{X}, G)$$

of pointed sets.

Remark 3.7. The above double coset space is reminiscent of the classical adelic double coset space

$$\Sigma_{G,K} := G(K) \setminus G(\mathbb{A}_K) / \prod_{v \in \Omega_K} G(\mathcal{O}_v),$$

where K is a function field over a finite field with set of discrete valuations Ω_K , complete local rings \mathcal{O}_v , and adeles \mathbb{A}_K . This space is indeed also used in the study of G -torsors over K and of $\text{III}(K, G)$. See, for example, [Con10], Sections 1.2 and 1.3.

4 The case of rational groups

In [HHK09], certain local-global principles were shown to hold for connected rational linear algebraic groups. In this manuscript, we permit consideration of groups that are not necessarily connected. For an arbitrary linear algebraic group G over an infinite field k , we say that G is *rational* over k (or *k-rational*) if every connected component of G is a rational variety over k . It can be shown that over an algebraically closed field, every linear algebraic group is rational. Rationality is equivalent to the condition that the identity component G^0 is k -rational and every component has a k -point. This also implies that G/G^0 is a constant finite group. In particular, for the inverse system of fields considered in Section 3,

the hypothesis of Corollary 2.6 is satisfied for any F -rational linear algebraic group G , with $N = G^0$ (also using Corollary 3.4). Hence for a rational linear algebraic group G over F , we obtain a short exact sequence of pointed sets

$$1 \rightarrow \text{III}_{\mathcal{P}}(\widehat{X}, G^0) \rightarrow \text{III}_{\mathcal{P}}(\widehat{X}, G) \rightarrow \text{III}_{\mathcal{P}}(\widehat{X}, G/G^0) \rightarrow 1$$

induced from the corresponding cohomology sequence.

Moreover, Proposition 3.3 shows that \mathcal{P} as in Notation 3.2 satisfies the hypotheses made in Section 3 of [HHK09]. Theorem 3.7 of [HHK09] then implies that $\text{III}_{\mathcal{P}}(\widehat{X}, G^0) = 1$. Summing up the discussion, we find that the natural surjection $\text{III}_{\mathcal{P}}(\widehat{X}, G) \rightarrow \text{III}_{\mathcal{P}}(\widehat{X}, G/G^0)$ has trivial kernel for rational linear algebraic groups G .

Note that whereas $\text{III}_{\mathcal{P}}(\widehat{X}, G) \rightarrow \text{III}_{\mathcal{P}}(\widehat{X}, G/G^0)$ is in general just a map of pointed sets, it is a group homomorphism in the case that G is commutative, being the restriction to $\text{III}_{\mathcal{P}}(\widehat{X}, G)$ of the group homomorphism $H^1(F, G) \rightarrow H^1(F, G/G^0)$. Hence if the linear algebraic group G is rational and commutative, the above says that this map is an isomorphism of groups. But in general, a map of pointed sets can have trivial kernel without being injective.

Nevertheless, we show below that (if G is rational) the map $\text{III}_{\mathcal{P}}(\widehat{X}, G) \rightarrow \text{III}_{\mathcal{P}}(\widehat{X}, G/G^0)$ is indeed a bijection of pointed sets. This reduces the study of $\text{III}_{\mathcal{P}}(\widehat{X}, G)$ for rational linear algebraic groups to the case of finite constant groups (i.e. to the study of finite Galois extensions of the function field F).

We first prove a variant on Theorem 2.5 of [HHK09]. We recall the situation considered there:

Let \widehat{R}_0 be a complete discrete valuation ring with uniformizer t and field of fractions F_0 , and suppose \widehat{R}_0 contains a subring $T \subseteq \widehat{R}_0$ that is also a complete discrete valuation ring with uniformizer t . We let $|\cdot|$ be a norm defined on F_0 defined by the t -adic valuation. This induces a maximum norm on $\mathbb{A}_{F_0}^n$ in the usual way. Let F_1, F_2 be subfields of F_0 containing T .

Proposition 4.1. *In the above situation, assume there are t -adically complete T -submodules $V \subset F_1 \cap \widehat{R}_0$, $W \subset F_2 \cap \widehat{R}_0$ satisfying $V + W = \widehat{R}_0$. Assume moreover that F_1 is t -adically dense in F_0 . Let $f = (f_1, \dots, f_n) : \mathbb{A}_{F_0}^n \times \mathbb{A}_{F_0}^n \dashrightarrow \mathbb{A}_{F_0}^n$ be an F_0 -rational map that is defined on a Zariski open set $U \subseteq \mathbb{A}_{F_0}^n \times \mathbb{A}_{F_0}^n$ containing the origin $(0, 0)$. Suppose that for each i we may write (in multi-index notation)*

$$f_i = T_{1,i}(x) + T_{2,i}(y) + \sum_{|(\nu, \rho)| \geq 2} c_{\nu, \rho, i} x^\nu y^\rho, \quad (*)$$

for some elements $c_{\nu, \rho, i}$ in F_0 , where $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$, and where $T_j = (T_{j,1}, \dots, T_{j,n})$ is an automorphism of the vector space F_0^n for $j = 1, 2$. Then there is a real number $\varepsilon > 0$ such that for all $a \in \mathbb{A}^n(F_0)$ with $|a| \leq \varepsilon$, there exist $v \in F_1^n$ and $w \in F_2^n$ such that $(v, w) \in U(F_0)$ and $f(v, w) = a$.

Proof. Case 1: $T_1 = T_2$ is the identity transformation.

In this situation, the assertion was shown at [HHK09, Theorem 2.5]. The statement of that result had assumed that $f(u, 0) = u = f(0, u)$ whenever $(u, 0)$ (resp. $(0, u)$) is in U ,

rather than assuming $(*)$. But that hypothesis was used only in order to obtain $(*)$ and the equality $f(0,0) = 0$, which itself follows from $(*)$. This was done in the first paragraph of the proof of [HHK09, Theorem 2.5]. The remainder of the proof of [HHK09, Theorem 2.5] used only $(*)$, and so in our situation that argument carries over and the desired conclusion follows.

Case 2: T_1 is the identity and T_2 is arbitrary.

The group $G := \mathrm{GL}_{n,F}$ is rational and connected over $F := F_1 \cap F_2$ and F_1 is dense in F_0 . Hence Theorem 3.2 of [HHK09] applies, and asserts that for any $g \in \mathrm{GL}_n(F_0)$ there exist $g_1 \in \mathrm{GL}_n(F_1)$ and $g_2 \in \mathrm{GL}_n(F_2)$ such that $g = g_1 g_2$. Taking g to be the matrix associated to T_2 , we obtain invertible linear transformations S_i of F_i^n for $i = 1, 2$ such that $T_2 = S_1 S_2^{-1}$.

Consider the F_0 -rational map $\tilde{f} = S_1^{-1} \circ f \circ (S_1 \times S_2) : \mathbb{A}_{F_0}^n \times \mathbb{A}_{F_0}^n \dashrightarrow \mathbb{A}_{F_0}^n$. This is defined on the Zariski open set $\tilde{U} := (S_1 \times S_2)^{-1}(U)$, which contains the origin $(0,0)$. Write $\tilde{f} = (\tilde{f}_1, \dots, \tilde{f}_n)$. Since $\tilde{f}(0,0) = 0$, we may write $\tilde{f}_i \in F_0[[x_1, \dots, x_n, y_1, \dots, y_n]]$ as $\tilde{f}_i = L_{1,i}(x) + L_{2,i}(y) + \sum_{\nu, \rho} \tilde{c}_{\nu, \rho, i} x^\nu y^\rho$ for some $\tilde{c}_{\nu, \rho, i}$ in F_0 , where $L_{1,i}$ and $L_{2,i}$ are linear forms in x_1, \dots, x_n and y_1, \dots, y_n respectively, where the sum ranges over $|(\nu, \rho)| \geq 2$ and where $1 \leq i \leq n$.

Let $L_j := (L_{j,1}, \dots, L_{j,n})$ for $j = 1, 2$. Equating linear terms in the definition of \tilde{f} then yields the equality $S_1^{-1} \circ (\mathrm{id} + T_2) \circ (S_1 \times S_2) = L_1 + L_2$, where id is the identity map on x_1, \dots, x_n and T_2 is viewed as a map on y_1, \dots, y_n . That is, $L_1(x) + L_2(y) = S_1^{-1}(S_1(x) + T_2 S_2(y)) = x + y$. We then have

$$\tilde{f}_i = x_i + y_i + \sum_{|(\nu, \rho)| \geq 2} \tilde{c}_{\nu, \rho, i} x^\nu y^\rho$$

for each i . Hence Case 1 applies, and there is a real number $\tilde{\varepsilon} > 0$ as in the assertion of the theorem. Since the linear map S_1 is continuous, there is an $\varepsilon > 0$ such that if $|a| \leq \varepsilon$ then $|S_1^{-1}(a)| \leq \tilde{\varepsilon}$. Now suppose that $a \in \mathbb{A}^n(F_0)$ with $|a| \leq \varepsilon$, and let $\tilde{a} := S_1^{-1}(a)$. By the assertion of the theorem in Case 1, there exist $\tilde{v} \in F_1^n$ and $\tilde{w} \in F_2^n$ such that $(\tilde{v}, \tilde{w}) \in \tilde{U}(F_0)$ and $\tilde{f}(\tilde{v}, \tilde{w}) = \tilde{a}$.

Let $v = S_1(\tilde{v}) \in F_1^n$ and $w = S_2(\tilde{w}) \in F_2^n$. Thus $(v, w) = (S_1 \times S_2)(\tilde{v}, \tilde{w}) \in U$. The equality $f = S_1 \circ \tilde{f} \circ (S_1 \times S_2)^{-1}$ then yields $f(v, w) = S_1(\tilde{f}(\tilde{v}, \tilde{w})) = S_1(\tilde{a}) = a$. Thus ε has the required properties. This concludes the proof in Case 2.

Case 3: General case.

Let $\tilde{f} = T_1^{-1} \circ f : \mathbb{A}_{F_0}^n \times \mathbb{A}_{F_0}^n \dashrightarrow \mathbb{A}_{F_0}^n$. Thus \tilde{f} is defined on U . Write $\tilde{f} = (\tilde{f}_1, \dots, \tilde{f}_n)$ and $S = (S_1, \dots, S_n) := T_1^{-1} T_2$. Then for each i we may write $\tilde{f}_i = x_i + S_i(y) + \sum_{|(\nu, \rho)| \geq 2} \tilde{c}_{\nu, \rho, i} x^\nu y^\rho$ for some $\tilde{c}_{\nu, \rho, i}$ in F_0 . So by Case 2, there is a real number $\tilde{\varepsilon} > 0$ as in the statement of the theorem. Since T_1^{-1} is continuous, there is an $\varepsilon > 0$ such that if $|a| \leq \varepsilon$ then $|T_1^{-1}(a)| \leq \tilde{\varepsilon}$. Thus if $a \in \mathbb{A}^n(F_0)$ satisfies $|a| \leq \varepsilon$ then by the assertion of the theorem (in Case 2), there exist $v \in F_1^n$ and $w \in F_2^n$ such that $(v, w) \in U(F_0)$ and $\tilde{f}(v, w) = T_1^{-1}(a)$, and hence $f(v, w) = a$. \square

We now return to the situation of Notation 3.2. Let G be a rational linear algebraic group over F , and let G^0 be its identity component.

Theorem 4.2. *Under Notation 3.2, if G is a rational linear algebraic group over F , then the natural map $\text{III}_{\mathcal{P}}(\widehat{X}, G) \rightarrow \text{III}_{\mathcal{P}}(\widehat{X}, G/G^0)$ is a bijection.*

Proof. Write $\bar{G} := G/G^0$; this is a finite constant group by the rationality assumption. By Corollary 3.4 and the rationality of G , the hypotheses of Proposition 2.7 are satisfied. Hence it suffices to show that for any two elements $(g_{\varphi}), (\tilde{g}_{\varphi}) \in \prod_{\mathcal{B}} G(F_{\varphi})$ that map to the same element in $\prod_{\mathcal{B}} \bar{G}$, there exist elements $g_{\xi} \in G(F_{\xi})$ for all $\xi \in \mathcal{P} \cup \mathcal{U}$ such that $g_{\varphi} = g_U^{-1} \tilde{g}_{\varphi} g_P$ whenever φ is a branch at P lying on the closure of U . Let $(h_{\varphi}) := (g_{\varphi}^{-1} \tilde{g}_{\varphi})$; this lies in $\prod_{\mathcal{B}} G^0(F_{\varphi})$.

By Proposition 3.3, there is a finite morphism $\widehat{X} \rightarrow \mathbb{P}_T^1$ such that $\mathcal{P} = f^{-1}(\infty)$. Write F' for the function field $K(x)$ of \mathbb{P}_T^1 , and let $d = [F : F']$. As in Notation 3.1 for \mathbb{P}_T^1 , we may consider rings and fields associated to ∞ , \mathbb{A}_k^1 , and to the unique branch on \mathbb{P}_k^1 at infinity; we call these $\widehat{R}_1, F_1, \widehat{R}_2, F_2, \widehat{R}_0, F_0$, respectively. Let G' be the Weil restriction $R_{F/F'}(G)$ of G (see, for example, [BLR90], Section 7.6). By the defining property of the Weil restriction, there is a natural isomorphism $G'(F_0) = G(F_0 \otimes_{F'} F)$, and the latter equals $\prod_{\mathcal{B}} G(F_{\varphi})$ by [HH10], Lemma 6.2(a). Similarly, $G'(F_1) = \prod_{\mathcal{P}} G(F_P)$ and $G'(F_2) = \prod_{\mathcal{U}} G(F_U)$. Under these identifications, write $(g_{\varphi}) =: g \in G'(F_0)$, $(\tilde{g}_{\varphi}) =: \tilde{g} \in G'(F_0)$, and $(h_{\varphi}) =: h \in G^0(F_0)$. Since G^0 is (geometrically) connected and smooth, so is $R_{F/F'}(G^0)$ ([Oes84], Proposition A.3.7; [CGP10], Proposition A.5.9). Thus $R_{F/F'}(G^0)$ is the identity component of G' , and moreover it is rational.

Consider the map $\gamma : G^0(F_0) \times G^0(F_0) \rightarrow G^0(F_0)$ by $(g_1, g_2) \mapsto g^{-1} g_2 g g_1^{-1}$. We claim that h is the image under γ of a pair $(g_1, g_2) \in G^0(F_1) \times G^0(F_2)$. Assuming this, $g^{-1} \tilde{g} = h = g^{-1} g_2 g g_1^{-1}$; or equivalently, $g = g_2^{-1} \tilde{g} g_1$. Viewing this again as an equation in $\prod_{\mathcal{B}} G(F_{\varphi})$ then yields the desired elements.

It remains to prove the claim. Since G^0 is rational, there is an F -isomorphism $\phi : U' \rightarrow U$ from a connected open neighborhood of the identity in G' to an open neighborhood of the origin in $\mathbb{A}_{F'}^n$ (for a suitable n). The map ϕ carries γ to an F_0 -rational map $f : \mathbb{A}_{F_0}^n \times \mathbb{A}_{F_0}^n \dashrightarrow \mathbb{A}_{F_0}^n$ that is defined on a Zariski open set $U \subseteq \mathbb{A}_{F_0}^n \times \mathbb{A}_{F_0}^n$ containing the origin $(0, 0)$. That is, $f \circ (\phi \times \phi)$ agrees with $\phi \circ \gamma$ as a rational map. Moreover $f(0, 0) = 0$. So as an element of $F_0[[x_1, \dots, x_n, y_1, \dots, y_n]]$, $f = (f_1, \dots, f_n)$ has the form $(*)$ of Proposition 4.1, with T_i being a linear transformation of the vector space F_0^n , corresponding to a matrix $A_i \in \text{Mat}_n(F_0)$. Here (A_1, A_2) is the Jacobian matrix of f at the origin, and A_1 (resp. A_2) is the Jacobian matrix of the restriction of f to $y_1 = \dots = y_n = 0$ (resp. to $x_1 = \dots = x_n = 0$). The matrices A_1, A_2 are invertible, since the restrictions $\gamma(\cdot, 1)$ and $\gamma(1, \cdot)$ are invertible; and hence T_i is an automorphism of F_0^n for $i = 1, 2$. Let $V = \widehat{R}_1$ and $W = \widehat{R}_2$. Since every element of $\widehat{R}_0/(t) = k((x^{-1}))$ is the sum of elements of $\widehat{R}_1/(t) = k[[x^{-1}]]$ and $\widehat{R}_2/(t) = k[[x]]$, a standard induction argument shows that $V + W = \widehat{R}_0$. So the hypotheses of Proposition 4.1 are satisfied. By the conclusion of that proposition, there is an $\varepsilon > 0$ such that if $a \in \mathbb{A}_{F_0}^n$ satisfies $|a| \leq \varepsilon$ then $a = f(v, w)$ for some $v \in F_1^n$ and $w \in F_2^n$.

Now F_1 is dense in F_0 (see the proof of [HHK09], Theorem 3.4). Hence we may apply Lemma 3.1 of [HHK09] to the rational group G^0 and the field F_0 to obtain an element $h_1 \in G^0(F_1)$ such that $|\phi(hh_1)| \leq \varepsilon$ (for details see the last paragraph of the proof of [HHK09], Theorem 3.2). Thus $\phi(hh_1) = f(v, w)$ for some $v \in F_1^n$ and some $w \in F_2^n$. Applying ϕ^{-1} , we

obtain the conclusion that hh_1 is the image under γ of a pair $(g'_1, g_2) \in (G')^0(F_1) \times (G')^0(F_2)$, i.e., $hh_1 = g^{-1}g_2g(g'_1)^{-1}$. But then $h = g^{-1}g_2g(g'_1)^{-1}h_1^{-1}$, or equivalently, $h = \gamma(g_1, g_2)$ for $g_1 := h_1g'_1$. This finishes the proof. \square

5 Split covers

In this section we consider a different and more canonical local-global map, with respect to *all* points of the closed fiber (including the generic points of components) of a curve \widehat{X} over a complete discrete valuation ring T , as in Notation 3.1.

We define

$$\text{III}_0(\widehat{X}, G) := \ker(H^1(F, G) \rightarrow \prod_{P \in X} H^1(F_P, G))$$

for a linear algebraic group G over our field F . For finite subsets $\mathcal{P} \subseteq \mathcal{P}'$ of the closed fiber X as in Notation 3.2, we have the containments

$$\text{III}_{\mathcal{P}}(\widehat{X}, G) \subseteq \text{III}_{\mathcal{P}'}(\widehat{X}, G) \subseteq \text{III}_0(\widehat{X}, G) \subseteq H^1(F, G).$$

Namely, if $P \in U$ then $F_U \subset F_P$, hence any torsor with an F_U -point has an F_P -point; and if $U' \subseteq U$ then similarly a torsor with an $F_{U'}$ -point has an F_U -point.

Below we show that for G rational, $\text{III}_0(\widehat{X}, G)$ equals $\text{III}_{\mathcal{P}}(\widehat{X}, G)$ (for any choice of \mathcal{P}), and we give its structure in terms of a certain quotient $\pi_1^{\text{split}}(\widehat{X})$ of the étale fundamental group of \widehat{X} (Theorem 5.10).

The study of $\text{III}_0(\widehat{X}, G)$ relies on the study of split covers. We will say that a finite morphism $h : \widehat{Y} \rightarrow \widehat{X}$ of normal projective T -curves is *split* over a point P of \widehat{X} if $\widehat{Y} \times_{\widehat{X}} P$ is a disjoint union of copies of P . In this case h is étale over P since the fiber $\widehat{Y} \times_{\widehat{X}} P$ is reduced and separable (in fact trivial) over P . More generally, given a morphism $S \rightarrow \widehat{X}$, we will similarly say that h is *split* over S (or over A , in case $S = \text{Spec}(A)$) if $\widehat{Y} \times_{\widehat{X}} S$ is a disjoint union of copies of S . We say that h is a *split cover* if it is split over every point P of \widehat{X} except possibly the generic point. Thus every split cover of \widehat{X} is étale. The inverse limit of the Galois groups of Galois split covers of \widehat{X} will be denoted by $\pi_1^{\text{split}}(\widehat{X})$; this is a quotient of the étale fundamental group $\pi_1(\widehat{X})$. Thus if we choose a base point for \widehat{X} , then for any finite constant group G , the pointed G -Galois split covers of \widehat{X} are classified by $\text{Hom}(\pi_1^{\text{split}}(\widehat{X}), G)$; and the (unpointed) G -Galois split covers of \widehat{X} are classified by $\text{Hom}(\pi_1^{\text{split}}(\widehat{X}), G)/\sim$, where the equivalence relation is given by conjugation by G . Here $\text{Hom}(\pi_1^{\text{split}}(\widehat{X}), G)/\sim$ is viewed as a subset of $\text{Hom}(\text{Gal}(F^{\text{sep}}/F), G)/\sim = H^1(F, G)$, which classifies the G -Galois field extensions of F . If G is abelian, then conjugation is trivial. Moreover, $H^1(F, G)$ is a group in that case, and $\text{Hom}(\pi_1^{\text{split}}(\widehat{X}), G)$ is a subgroup.

Observe that if $h : \widehat{Y} \rightarrow \widehat{X}$ is a finite morphism of T -curves and P is a point of \widehat{X} , then h is split over P if and only if h is split over the ring \widehat{R}_P , by Hensel's Lemma, which applies since being split implies being étale. If moreover \widehat{Y} is normal then these conditions are also equivalent to h being split over the field F_P , since in that case the fiber over \widehat{R}_P is the

normalization of \widehat{R}_P in the fiber over F_P . For $U \in \mathcal{U}$ as in Notation 3.2, the corresponding equivalences hold for U, \widehat{R}_U, F_U ; this uses [HHK09, Lemma 4.5] instead of Hensel's Lemma.

Proposition 5.1. *Suppose that $h : \widehat{Y} \rightarrow \widehat{X}$ is a finite morphism of normal projective T -curves, with \widehat{X} connected. Then h is a split cover if and only if it is split over F_P for every (not necessarily closed) point P of the closed fiber X .*

Proof. The forward direction follows from the above observation, which also shows that if h is split over every F_P then it is split over every point P of X .

To complete the proof of the converse, it remains to show that h is split at each codimension one point of \widehat{X} that is not supported on X . Such a point is a closed point of the general fiber of \widehat{X} , with residue field a finite extension K' of K . Its closure Z in \widehat{X} is of the form $\text{Spec}(T')$ for some finite extension T' of T , since \widehat{X} is proper over T . So T' is a t -adically complete one-dimensional domain (but not necessarily normal). The maximal ideal of T' corresponds to a closed point P of X ; and the maximal ideal of the local ring R_P is the radical of the ideal $J := (I, t)$, where I is the ideal of Z in $\text{Spec}(R_P)$. Thus \widehat{R}_P is the J -adic completion of R_P , and so the natural map $R_P/I \rightarrow \widehat{R}_P/I\widehat{R}_P$ is an isomorphism (because $T' = R_P/I$ is t -adically complete). Since h is split over \widehat{R}_P , it follows that it is split over R_P/I , hence over the codimension one point, as desired. \square

Remark 5.2. (a) Proposition 5.1 shows that there is a natural bijection between the set of split covers of \widehat{X} and the set of finite morphisms to the closed fiber X that are split over every point. This follows from Hensel's Lemma and the fact that every étale cover of X lifts uniquely to an étale cover of \widehat{X} ([Gro71], Theorem X.2.1).

(b) In Proposition 5.1, if the residue field k of T is finite, then Chebotarev's Density Theorem applies. Hence $h : \widehat{Y} \rightarrow \widehat{X}$ is a split cover if and only if it is split over every *closed* point of \widehat{X} (i.e. of X). A cover satisfying this latter splitness condition is referred to in [Sai85], Definition II.2.1, as a *c.s. cover*. If the residue field k is not assumed to be finite (e.g. if it is algebraically closed), then that condition is in general weaker than being a split cover. Compare Corollary 6.4 below to Proposition 2.2 and Theorem 2.4 of [Sai85], Section II (where k was assumed to be finite).

(c) In the case of Galois covers, the condition of being split is equivalent to being open in the Nisnevich (or “completely decomposed”) topology. See [Nis89] or [Ras95], Section 1.7.

Corollary 5.3. *Suppose that $h : \widehat{Y} \rightarrow \widehat{X}$ is a finite morphism of projective T -curves, with \widehat{X} regular and connected, and with \widehat{Y} normal. Then h is a split cover if and only if it is split over every codimension one point of \widehat{X} .*

Proof. The forward direction is trivial. For the converse, we want to show that h is split over every closed point. By Purity of Branch Locus, h is étale; and hence every fiber is reduced and has separable residue field extension. Any closed point Q of \widehat{X} has codimension two, and by regularity there is a system of local uniformizing parameters at Q . Consider the zero

locus of one of these parameters, and let Z be the irreducible component of this locus that passes through Q . Then h splits over the generic point of the regular curve Z by hypothesis, since this point has codimension one. Hence there is no residue field extension over the closed point Q and thus h is split there. The result then follows from Proposition 5.1. \square

We now return to the situation of Notation 3.2. However, when choosing a finite subset \mathcal{P} of the closed fiber X , we now strengthen our hypotheses slightly:

Hypothesis 5.4. Under Notation 3.2, assume in addition that the finite set $\mathcal{P} \subset X$ contains all the closed points at which X is not unibranched. In particular, it contains all points where an irreducible component intersects itself.

We then have the following analog of Proposition 5.1:

Corollary 5.5. *In the situation of Hypothesis 5.4, suppose that $h : \hat{Y} \rightarrow \hat{X}$ is a finite morphism of normal projective T -curves, with \hat{X} connected. Then h is a split cover if and only if it is split over every F_U (for $U \in \mathcal{U}$) and every F_P (for $P \in \mathcal{P}$).*

Proof. In the forward direction, h is split over the generic point of each $U \in \mathcal{U}$ (this being a codimension one point of \hat{X}), and h is étale. Let Y be the closed fiber of \hat{Y} ; and for any choice of $U \in \mathcal{U}$ let $V \subseteq Y$ be a connected component of $h^{-1}(U)$. If V is reducible, then each irreducible component of V must meet some other component at some closed point Q , which is thus not unibranched. But since h is étale, $h(Q) \in U$ is also not unibranched. But \mathcal{P} contains all non-unibranched points by Hypothesis 5.4, and so such a point cannot lie in U . This is a contradiction. So actually V is irreducible, and has a unique generic point. By splitness, the residue field at that generic point is a trivial field extension of the residue field at the generic point of U . Thus $V \rightarrow U$ is a connected finite étale cover of degree one, and hence an isomorphism. This shows that $h^{-1}(U)$ is a disjoint union of copies of U ; i.e. h splits over U . By the observation before Proposition 5.1, h is split over F_U . Also, h is split over each F_P by Proposition 5.1.

In the converse direction, since h is split over each F_U it is split over each \hat{R}_U (since \hat{Y} is normal), and hence over \hat{R}_Q for every (not necessarily closed) point $Q \in U$. But every point of X either lies in \mathcal{P} or in some $U \in \mathcal{U}$. So h is split by Proposition 5.1. \square

Notice that Proposition 5.5 fails without the extra assumption in Hypothesis 5.4. For example, take a non-trivial split cover of a rational nodal curve, if the nodal point is not in the set \mathcal{P} .

Let G be a finite constant group. Given a G -Galois étale cover $\hat{Y} \rightarrow \hat{X}$ with \hat{X} connected, its generic fiber is a G -torsor $\mathcal{T} = \text{Spec}(A)$ over the function field F of \hat{X} , and \hat{Y} is the normalization of \hat{X} in A . Moreover $\hat{Y} \rightarrow \hat{X}$ is split over a field extension E/F if and only if the G -torsor \mathcal{T} becomes trivial over E . Conversely, given a G -torsor $\mathcal{T} = \text{Spec}(A)$ over F , the normalization $\hat{Y} \rightarrow \hat{X}$ of \hat{X} in A is a finite morphism of normal projective T -curves, with a G -action that extends that of its generic fiber \mathcal{T} and satisfies $\hat{Y}/G = \hat{X}$. Thus Proposition 5.1 and Corollary 5.5 yield:

Corollary 5.6. *Let \widehat{X} be a connected normal projective T -curve with function field F . Then for any finite constant group G , and any $\mathcal{P} \subset X$ satisfying Hypothesis 5.4, the subsets $\mathrm{Hom}(\pi_1^{\mathrm{split}}(\widehat{X}), G)/\sim$, $\mathrm{III}_0(\widehat{X}, G)$, and $\mathrm{III}_{\mathcal{P}}(\widehat{X}, G)$ of $H^1(F, G)$ each classify the G -Galois finite split covers of \widehat{X} , and are thus equal.*

Our aim is to extend the statements of Corollary 5.6 to arbitrary rational linear algebraic groups. Using the results of Section 4, we immediately obtain:

Corollary 5.7. *Under Hypothesis 5.4, if G is a rational linear algebraic group then $\mathrm{III}_{\mathcal{P}}(\widehat{X}, G)$ is in natural bijection with the pointed set $\mathrm{Hom}(\pi_1^{\mathrm{split}}(\widehat{X}), G/G^0)/\sim$. Hence $\mathrm{III}_{\mathcal{P}}(\widehat{X}, G)$ is independent of the choice of \mathcal{P} .*

Proof. By Theorem 4.2, the natural map $\mathrm{III}_{\mathcal{P}}(\widehat{X}, G) \rightarrow \mathrm{III}_{\mathcal{P}}(\widehat{X}, G/G^0)$ is a bijection. Since G is rational, its quotient G/G^0 is a constant finite group. Corollary 5.6 then implies that $\mathrm{III}_{\mathcal{P}}(\widehat{X}, G)$ is in natural bijection with $\mathrm{Hom}(\pi_1^{\mathrm{split}}(\widehat{X}), G/G^0)/\sim$.

The second assertion follows from the fact that if $\mathcal{P} \subseteq \mathcal{P}'$ then the bijections with $\mathrm{Hom}(\pi_1^{\mathrm{split}}(\widehat{X}), G/G^0)/\sim$ are compatible with the inclusion $\mathrm{III}_{\mathcal{P}}(\widehat{X}, G) \subseteq \mathrm{III}_{\mathcal{P}'}(\widehat{X}, G)$. \square

To extend this result to $\mathrm{III}_0(\widehat{X}, G)$, thus generalizing Corollary 5.6, we first prove two preliminary results.

Proposition 5.8. *Let \widehat{X} be a normal projective curve over a complete discrete valuation ring T , and let η be the generic point of an irreducible component X_0 of the closed fiber X . Let H be a variety over F . If $H(F_{\eta})$ is non-empty, then so is $H(F_U)$ for some non-empty affine open subset $U \subset X_0$ that does not meet any other irreducible component of X .*

Proof. Since \widehat{X} is normal and η is a point of codimension one, its local ring R_{η} is a discrete valuation ring, say with uniformizer s . The given F_{η} -point on H lies on an affine open subset H' of H . Here H' is given in affine n -space by polynomials $f_j(x_1, \dots, x_n) \in F[x_1, \dots, x_n]$, for $j = 1, \dots, m$. Let the coordinates of the F_{η} -point be $(\hat{a}_1, \dots, \hat{a}_n)$, with $\hat{a}_i \in F_{\eta}$. For some integer c , the elements $s^c \hat{a}_i$ all lie in the complete local ring \widehat{R}_{η} . Setting $g_j(x_1, \dots, x_n) = f_j(s^{-c}x_1, \dots, s^{-c}x_n)$ and replacing the polynomials f_j by the polynomials g_j , we may assume that each \hat{a}_i lies in \widehat{R}_{η} . Also, multiplying the polynomials g_j by appropriate powers of s , we may assume that each g_j lies in $R_{\eta}[x_1, \dots, x_n]$.

Since T is a complete discrete valuation ring, it is excellent by [Gro65, Scholie 7.8.3(iii)]; and since \widehat{X} is of finite type over T , it follows from [Gro65, Scholie 7.8.3(ii)] that the local ring R_{η} is also excellent. Hence the Artin Approximation Theorem ([Art69], Theorem 1.10) applies. Thus there exist elements a_1, \dots, a_n of the henselization \tilde{R}_{η} of R_{η} that satisfy the polynomials g_j and are congruent to the elements \hat{a}_i modulo s . Since the henselization is a limit of étale neighborhoods, the elements a_i all lie in some étale neighborhood of η , i.e. in some étale R_{η} -algebra S that is contained in \tilde{R}_{η} . This containment defines a section of $\mathrm{Spec}(S) \rightarrow \widehat{X}$ over η . Since R_{η} is the local ring of \widehat{X} at η , and since $\mathrm{Spec}(S) \rightarrow \widehat{X}$ is of finite type, there exists an affine neighborhood $\mathrm{Spec}(A)$ of η in \widehat{X} together with an étale A -algebra $S_0 \subset S$ such that $S_0 \otimes_A R_{\eta}$ is isomorphic to S as an R_{η} -algebra, and which contains

a_1, \dots, a_n . The section over η defines a rational section over X_0 and hence a section over some affine open neighborhood $U \subset X$ of η in X . After shrinking U we may assume that it is contained in the subset of X_0 consisting of points that do not meet any other component of X . After shrinking $\text{Spec}(A)$ we may assume that U is its closed fiber.

Since $\text{Spec}(S_0) \rightarrow \text{Spec}(A)$ is étale, [HHK09, Lemma 4.5] implies that the section over U extends to a section over $\text{Spec}(\widehat{R}_U)$. That is, the inclusion of S_0 into $\tilde{R}_\eta \subset \widehat{R}_\eta$ defines an inclusion of S_0 into \widehat{R}_U , and hence into F_U . The S_0 -valued point (a_1, \dots, a_n) of $H' \subseteq H$ thus determines an F_U -point of H , as desired. \square

Corollary 5.9. *Under Notation 3.1, let G be any linear algebraic group over the function field F . Then $\text{III}_0(\widehat{X}, G) = \bigcup_{\mathcal{P}} \text{III}_{\mathcal{P}}(\widehat{X}, G)$, where the union ranges over all finite sets \mathcal{P} of closed points of \widehat{X} that satisfy Hypothesis 5.4.*

Proof. Since each $\text{III}_{\mathcal{P}}(\widehat{X}, G)$ is contained in $\text{III}_0(\widehat{X}, G)$, it suffices to show that every element of $\text{III}_0(\widehat{X}, G)$ lies in some $\text{III}_{\mathcal{P}}(\widehat{X}, G)$. So consider a torsor representing an element of $\text{III}_0(\widehat{X}, G)$. Let η be the generic point of an irreducible component X_0 of X . Then the given torsor is trivial over F_η and so has an F_η -point. By Proposition 5.8, the torsor has an F_U -point for some affine Zariski open subset $U \subset X_0$ that does not meet any other irreducible component of X . Hence the given torsor is trivial over F_U . In this way we obtain such U on each irreducible component X_0 of X . Take \mathcal{U} to be the set of these open subsets U , indexed by the irreducible components of X ; and take \mathcal{P} to be the complement in X of the union of the sets U . Then the given torsor represents an element in $\text{III}_{\mathcal{P}}(\widehat{X}, G)$. \square

Theorem 5.10. *Let F be a one variable function field over the field of fractions of a complete discrete valuation ring T , and let \widehat{X} be a normal model for F over T . Let G be a rational linear algebraic group defined over F . Then*

$$\text{III}_0(\widehat{X}, G) = \text{III}_{\mathcal{P}}(\widehat{X}, G)$$

for any set \mathcal{P} that satisfies Hypothesis 5.4. Both are in natural bijection, as pointed sets, with $\text{Hom}(\pi_1^{\text{split}}(\widehat{X}), G/G^0)/\sim$. In particular, the natural map of pointed set $\text{III}_0(\widehat{X}, G) \rightarrow \text{III}_0(\widehat{X}, G/G^0)$ is a bijection.

Proof. Since $\text{III}_{\mathcal{P}}(\widehat{X}, G)$ is independent of the choice of \mathcal{P} by Corollary 5.7, the first assertion is immediate from Corollary 5.9. The second and third statements then follow by Corollary 5.7 and Theorem 4.2. \square

6 The Reduction Graph

Split covers of a T -curve \widehat{X} can be understood in terms of a combinatorial object, viz. the reduction graph Γ associated to the closed fiber of \widehat{X} . Using this, we obtain a more explicit description of $\text{III}_0(\widehat{X}, G) = \text{III}_{\mathcal{P}}(\widehat{X}, G)$ in the case of rational linear algebraic groups. This description (Corollary 6.5) shows that $\text{III}_0(\widehat{X}, G)$ is finite, and it provides a necessary and

sufficient condition for it to vanish and thus for the corresponding local-global principle to hold (still under the assumption that G is rational). This condition is given in terms of the rational group G and the reduction graph Γ .

We saw in Section 3 how a finite subset of points \mathcal{P} as in Notation 3.2 yields a factorization inverse system of fields, and in Section 2 how such systems relate to (multi-)graphs. In the situation of Notation 3.2, the associated graph $\Gamma = \Gamma(\widehat{X}, \mathcal{P})$ has vertices consisting of the elements of $\mathcal{P} \cup \mathcal{U}$, and edges consisting of the elements of \mathcal{B} . The vertices of an edge \wp are P and U , where \wp is a branch of X at $P \in \mathcal{P}$ lying on the closure of $U \in \mathcal{U}$. Thus each edge connects a vertex in \mathcal{P} to a vertex in \mathcal{U} . Graphs with this property (namely that the vertices are partitioned into two disjoint sets such that each edge has a vertex in each set) are called *bipartite*. We may regard the graph either as a combinatorial object or (as we will do more often) as a topological object — viz. a one-dimensional simplicial complex. Note that the graph is connected as a topological space because X is connected.

Remark 6.1. (a) Classically, there is another (multi-)graph that is associated to the above situation in the special case that X has only ordinary double points, and \mathcal{P} is the set of double points. Namely, the edges are given by the points of \mathcal{P} ; the vertices are given by the irreducible components of the closed fiber; and the vertices of a given edge $P \in \mathcal{P}$ are the components of X that pass through the point P . (See [DM69, p. 86], [Liu02, Definition 3.17], and [Sai85, Definition II.2.3].) In this special case, it is easy to see that this graph is homotopic, as a topological space, to our graph Γ . (In fact, our graph is the barycentric subdivision of the classical reduction graph.) Hence they have the same fundamental groups and cohomology.

- (b) If $(\widehat{X}, \mathcal{P})$ is as in Hypothesis 5.4 and if \mathcal{Q} is a finite set of closed points that contains \mathcal{P} , then the graphs $\Gamma(\widehat{X}, \mathcal{P})$ and $\Gamma(\widehat{X}, \mathcal{Q})$ are homotopic, with the latter differing from the former by having additional terminal vertices and edges corresponding to additional points and their branches. (This requires the additional assumption of Hypothesis 5.4, since self-intersecting components can introduce loops into the graph.) Moreover if $\widehat{X}' \rightarrow \widehat{X}$ is a blow-up of \widehat{X} at a regular point of X , and if $\mathcal{P}' \subset \widehat{X}'$ is a finite set of closed points over \mathcal{P} that contains the non-unibranched points of the closed fiber, then the graphs $\Gamma(\widehat{X}, \mathcal{P})$ and $\Gamma(\widehat{X}', \mathcal{P}')$ are homotopic.

Under Hypothesis 5.4, if $h : \widehat{Y} \rightarrow \widehat{X}$ is a split cover with closed fiber Y , then we may consider the finite set $h^{-1}(\mathcal{P}) \subset Y$. In this situation, $(\widehat{Y}, h^{-1}(\mathcal{P}))$ also satisfies Hypothesis 5.4, and we obtain a graph $\Gamma' := \Gamma(\widehat{Y}, h^{-1}(\mathcal{P}))$. There is a natural map of graphs $\Gamma' \rightarrow \Gamma$, and this is a finite covering space. Conversely, every finite covering space of Γ has the structure of a graph, by taking as the set of vertices the inverse image of the vertices of Γ .

Proposition 6.2. *With $(\widehat{X}, \mathcal{P})$ as in Hypothesis 5.4, the above association defines a lattice isomorphism between (connected) split covers of \widehat{X} and (connected) finite covering spaces of $\Gamma = \Gamma(\widehat{X}, \mathcal{P})$, which preserves the degree of the covers.*

Proof. By Proposition 3.3, there is a finite morphism $\widehat{X} \rightarrow \mathbb{P}_T^1$ such that $\mathcal{P} = f^{-1}(\infty)$. So by [HH10, Theorem 7.1(iv)] in the context of [HH10, Theorem 6.4], giving a finite separable

commutative F -algebra A is equivalent to giving its base change to each of the fields F_P (for $P \in \mathcal{P}$) and F_U (for $U \in \mathcal{U}$) together with isomorphisms between the algebras that they induce over the fields F_φ (for $\varphi \in \mathcal{B}$ a branch at P lying on the closure of U). But giving such an F -algebra A is also equivalent to giving a normal projective T -curve \widehat{Y} together with a finite generically separable morphism to \widehat{X} (where A is the generic fiber of \widehat{Y} , and \widehat{Y} is the normalization of \widehat{X} in A). Thus by Proposition 5.5, giving a split cover of \widehat{X} of degree n is equivalent to giving an F_φ -isomorphism of $(\prod_{i=1}^n F_\xi) \otimes_{F_\xi} F_\varphi$ with $\prod_{i=1}^n F_\varphi$ for each pair $\xi \in \mathcal{P} \cup \mathcal{U}$ and $\varphi \in \mathcal{B}$ such that φ is a branch at ξ (if $\xi \in \mathcal{P}$) or on the closure of ξ (if $\xi \in \mathcal{U}$). That is, we are given such an isomorphism for each pair (ξ, φ) such that ξ is a vertex of the edge φ in the reduction graph Γ . This is the same as giving a family of permutations $\sigma_{\xi, \varphi} \in S_n$ indexed by such pairs, up to equivalence coming from renumbering the n copies of each F_ξ and of each F_φ . That is, a family of permutations $\{\sigma_{\xi, \varphi}\}_{\xi, \varphi}$ is equivalent to the family $\{\tau_\xi^{-1} \sigma_{\xi, \varphi} \tau_\varphi\}_{\xi, \varphi}$ for each choice of permutations $\tau_\xi, \tau_\varphi \in S_n$ indexed by the elements $\xi \in \mathcal{P} \cup \mathcal{U}$ and $\varphi \in \mathcal{B}$.

Meanwhile, to give a covering space of Γ having degree n is to give a graph Γ' whose vertex set is $(\mathcal{P} \cup \mathcal{U}) \times \{1, \dots, n\}$ and whose edge set is $\mathcal{B} \times \{1, \dots, n\}$. Here Γ' is determined by specifying, for each edge (φ, i) , the corresponding vertices (P, j) and (U, ℓ) , where P, U correspond to the vertices of the edge φ in Γ , and where $1 \leq i, j, \ell \leq n$. For each such pair (φ, P) , the map $i \mapsto j$ is an element of S_n ; and similarly for (φ, U) and $i \mapsto \ell$. Again we have equivalence corresponding to renumbering the indices. Associating to each split cover of \widehat{X} the covering space of Γ with the same set of permutations thus gives a bijection between the split covers of \widehat{X} and the covering spaces of Γ . This is in fact the same map $\widehat{Y} \mapsto \Gamma'$ given before the proposition, by the definition of the graph associated to the closed fiber.

A split cover $\widehat{Y} \rightarrow \widehat{X}$ of degree n factors through a split cover $\widehat{Y}' \rightarrow \widehat{X}$ of degree m if and only if for some representative families of permutations $\{\sigma_{\xi, \varphi}\}_{\xi, \varphi}$ and $\{\sigma'_{\xi, \varphi}\}_{\xi, \varphi}$ for these covers there is an n/m -to-one map $\phi : \{1, \dots, n\} \rightarrow \{1, \dots, m\}$ such that $\phi \sigma_{\xi, \varphi} = \sigma'_{\xi, \varphi} \phi$ for all (ξ, φ) . The same is true for the covering spaces of Γ . So the above bijection of lattices preserves the property of one cover dominating another. As a result, it preserves meet and join in the lattices, and so it is a lattice isomorphism.

Finally, a split cover of \widehat{X} , or a covering space of Γ , is connected if and only if the set of associated permutations is transitive. So the above bijection carries connected split covers to connected covering spaces. \square

Remark 6.3. Since Proposition 6.2 concerns finite covers, it could also have been proven using other forms of patching (e.g. formal patching), rather than using patching over fields. See the introductory comments in Section 7.2 of [HH10].

Recall that $\pi_1^{\text{split}}(\widehat{X})$ is the inverse limit of the Galois groups of Galois split covers of \widehat{X} as introduced in Section 5. Directly from Proposition 6.2, we obtain:

Corollary 6.4. *Under Hypothesis 5.4, $\pi_1^{\text{split}}(\widehat{X})$ is naturally isomorphic to the profinite completion of $\pi_1(\Gamma(\widehat{X}, \mathcal{P}))$. Thus it is a free profinite group on finitely many generators.*

Here the last assertion follows from the fact that the fundamental group of any graph is a free group on finitely many generators.

As a consequence of Corollary 6.4, $\pi_1^{\text{split}}(\widehat{X})$ is trivial if and only if the graph $\Gamma(\widehat{X}, \mathcal{P})$ is a tree. As another consequence, the fundamental group of this graph is independent of the choice of the set \mathcal{P} , and may be denoted by $\pi_1(\Gamma(\widehat{X}))$.

Corollary 6.5. *Let G be a rational linear algebraic group. Then $\text{III}_0(\widehat{X}, G)$ is finite, and is in natural bijection with the pointed set $\text{Hom}(\pi_1(\Gamma(\widehat{X})), G/G^0)/\sim$. It is trivial if and only if either G is connected or $\pi_1(\Gamma(\widehat{X})) = 1$. Moreover, for any \mathcal{P} as in Hypothesis 5.4, the same holds for $\text{III}_{\mathcal{P}}(\widehat{X}, G)$.*

Proof. Theorem 5.10 and Corollary 6.4 together provide the asserted natural bijection, using that $\text{Hom}(\Pi, H)$ is in natural bijection with $\text{Hom}(\widehat{\Pi}, H)$ for any finite group H and any discrete group Π with profinite completion $\widehat{\Pi}$. The set $\text{Hom}(\pi_1(\Gamma(\widehat{X})), G/G^0)/\sim$ is finite because G/G^0 is finite and $\pi_1(\Gamma(\widehat{X}))$ is finitely generated. This gives the first assertion.

The set $\text{Hom}(\pi_1(\Gamma(\widehat{X})), G/G^0)/\sim$ is trivial exactly when the set $\text{Hom}(\pi_1(\Gamma(\widehat{X})), G/G^0)$ is trivial, since the only element conjugate to the identity is itself. As observed above, $\pi_1(\Gamma(\widehat{X}))$ is a free group of finite rank. But the set of homomorphisms from such a free group to a finite (constant) group is trivial if and only if either the free group or the constant group is trivial. So the second assertion now follows from the first assertion. The last assertion follows again from Theorem 5.10. \square

Remark 6.6. In the special case that \widehat{X} is smooth over T , the reduction graph is trivial and so Corollary 6.5 says that $\text{III}_{\mathcal{P}}(\widehat{X}, G)$ is trivial (i.e. the associated local-global principle is satisfied) even if G is a disconnected rational group. The corresponding simultaneous factorization result also appeared in [HHK09, Theorem 3.4], where connectivity was not in fact needed or used.

7 Split Extensions

In order to relate $\text{III}_0(\widehat{X}, G)$ to local-global principles in terms of valuations in Section 8, we now restrict to the situation where the model \widehat{X} of F is regular (i.e. \widehat{X} is a connected regular projective curve over T with function field F). We call such a T -curve \widehat{X} a *regular model* of F . In this situation we give a further description of $\text{III}_0(\widehat{X}, G)$ for rational linear algebraic groups in terms of the Galois cohomology of F (Theorem 7.10), showing in particular that $\text{III}_0(\widehat{X}, G)$ depends just on the field F , not the choice of regular model. Our approach will rely on the notion of split extensions. We say that a finite extension E/F of fields is *split* if $E \otimes_F F_v$ is a product of copies of the completion F_v of F with respect to v , for every discrete valuation v of F (i.e. valuation on F with value group isomorphic to \mathbb{Z}). Let F^{split} be the union of all split extensions of F , inside a fixed algebraic closure of F .

Proposition 7.6 below gives a characterization of split extensions of F in terms of split covers of a regular model. For this, we need some preliminary results about valuations, which are also needed in the next section.

Proposition 7.1. *Let R be a complete local domain that is not a field, and let v be a discrete valuation on the fraction field of R . Then R is contained in the valuation ring of v .*

Proof. Since R is not a field, its maximal ideal \mathfrak{m} contains a non-zero element m . For any non-zero $r \in R$ and any positive integer i , the element $r^i m$ lies in \mathfrak{m} .

So by Hensel's Lemma, $1 + r^i m$ is an n -th power in R for every n that is not divisible by the residue characteristic of R . Thus $v(1 + r^i m)$ is divisible by arbitrarily large integers in the value group of v ; and so $v(1 + r^i m) = 0$. Thus $1 + r^i m$ lies in the valuation ring of v , and hence so does $r^i m$. Thus $iv(r) + v(m) = v(r^i m) \geq 0$ for every positive integer i . This implies that $v(r) \geq 0$ and hence r is in the valuation ring of v . \square

Corollary 7.2. *Let F be a field that contains a complete discrete valuation ring T , and let v be any discrete valuation on F . Then the valuation ring of v contains T .*

Proof. If the restriction of v to the fraction field K of T is trivial, then the valuation ring of v contains K and hence contains T . Alternatively, if the restriction of v to K is non-trivial, then this restriction is a discrete valuation on K , and the assertion follows from Proposition 7.1 by taking $R = T$. \square

Let Z be an integral scheme and let v be a discrete valuation on the function field F of Z , with valuation ring $R_v \subset F$. Recall that a point Q of Z is called the *center* of v if R_v contains the local ring $\mathcal{O}_{Z,Q}$ of Q on Z , and if the maximal ideal \mathfrak{m}_Q of $\mathcal{O}_{Z,Q}$ is the contraction of the maximal ideal \mathfrak{m}_v of R_v . This is equivalent to the condition that the morphism $\text{Spec}(F) \rightarrow \text{Spec}(\mathcal{O}_{Z,Q})$ factors through the morphism $\text{Spec}(F) \rightarrow \text{Spec}(R_v)$. Since such a point $Q \in Z$ is the image of the closed point of $\text{Spec}(R_v)$ under the natural morphism $\text{Spec}(R_v) \rightarrow Z$, the point Q is unique if it exists. Note that since the valuation v is non-trivial, R_v is strictly contained in F and hence so is $\mathcal{O}_{Z,Q}$; thus the center of v cannot be the generic point of Z .

Lemma 7.3. *Let T be a complete discrete valuation ring, and let \widehat{X} be a connected regular projective T -curve with function field F . Then every discrete valuation v on F has a center Q on \widehat{X} , where Q is not the generic point of \widehat{X} . Moreover the completion F_v contains the fraction field of the complete local ring $\widehat{\mathcal{O}}_{\widehat{X},Q}$ of \widehat{X} at Q .*

Proof. By Corollary 7.2, the valuation ring $R_v \subset F$ of v contains T ; and this inclusion corresponds to a morphism $\text{Spec}(R_v) \rightarrow \text{Spec}(T)$. Since \widehat{X} is proper over $\text{Spec}(T)$, it follows from the valuative criterion of properness [Hts77, Theorem II.4.7] that there is a morphism $j : \text{Spec}(R_v) \rightarrow \widehat{X}$ such that the following diagram commutes:

$$\begin{array}{ccc} \text{Spec}(F) & \xrightarrow{\quad} & \widehat{X} \\ \downarrow & \nearrow \text{dashed} & \downarrow \\ \text{Spec}(R_v) & \xrightarrow{\quad} & \text{Spec}(T) \end{array}$$

Thus $\text{Spec}(R_v) \rightarrow \widehat{X}$ and hence $\text{Spec}(F) \rightarrow \widehat{X}$ factors through $\text{Spec}(\mathcal{O}_{\widehat{X},Q})$, where $Q \in \widehat{X}$ is the image under j of the closed point of $\text{Spec}(R_v)$. So Q is the center of v (which is not the

generic point of \widehat{X} , as observed before the proposition). In particular, we have an inclusion $\mathcal{O}_{\widehat{X},Q} \subseteq R_v$, with the maximal ideal of R_v contracting to that of $\mathcal{O}_{\widehat{X},Q}$. This in turn yields an inclusion $\widehat{\mathcal{O}}_{\widehat{X},Q} \subseteq \widehat{R}_v$ between their completions; and taking fraction fields yields the desired inclusion. \square

The next result strengthens the last assertion of Lemma 7.3 to show that for each discrete valuation v , there is in fact a point P on the closed fiber X (rather than just on \widehat{X}) for which $F_P \subseteq F_v$. We first introduce some notation. If Q is a codimension one point of a noetherian connected regular scheme, then it defines a discrete valuation v on the function field F , whose ring of integers R_v is the local ring R_Q at the point Q ([Hts77], p. 130). The completion \widehat{R}_Q of R_Q with respect to its maximal ideal \mathfrak{m}_Q is the v -adic completion \widehat{R}_v of R_v , and the fraction field F_Q of \widehat{R}_Q is the fraction field of \widehat{R}_v , viz. the v -adic completion F_v of F .

Proposition 7.4. *Under the hypotheses of Lemma 7.3, for every discrete valuation v on F there is a point P on the closed fiber of \widehat{X} such that $F_P \subseteq F_v$.*

Proof. By Lemma 7.3, the valuation v has a center Q , which is not the generic point of \widehat{X} . If Q lies on the closed fiber X of \widehat{X} , then we may take $Q = P$. Otherwise, Q lies on the general fiber of \widehat{X} , which is a regular projective curve over the fraction field K of T . The point Q has codimension one, and v is defined by Q , so $F_Q = F_v$. The residue field at Q is a finite extension K' of K . Here K' is a complete discretely valued field, and is the function field of the closure of Q in \widehat{X} . Let $P \in X$ be the closed point of \widehat{X} corresponding to the maximal ideal of the valuation ring $T' \subset K'$. It remains to show that $F_P \subseteq F_Q$.

The point P is in the closure of Q , and so R_P is contained in R_Q . Since \widehat{X} is regular, R_P is a regular local ring. So the height one prime ideal $I \subset R_P$ defining Q is principal; say $I = (f)$. We claim that \widehat{R}_P , the completion of R_P with respect to its maximal ideal \mathfrak{m} , is also the I -adic completion of R_P . If this is shown, then the inclusion $R_P \subset R_Q$ induces an inclusion $\widehat{R}_P \subset \widehat{R}_Q$ on the I -adic completions, and hence induces the desired inclusion $F_P \subseteq F_Q$.

The ideal $\mathfrak{m} \subset R_P$ is the radical of the ideal (f, t) , where t is the uniformizer of T ; so the \mathfrak{m} -adic and (f, t) -adic topologies on R_P are the same. For any positive integer n , the ring R_P/I^n is t -adically complete because this ring is a finite module over the t -adically complete ring T . But a sequence in R_P/I^n is t -adically Cauchy if and only if it is (f, t) -adically Cauchy, because f is nilpotent in R_P/I^n . Hence R_P/I^n is (f, t) -adically complete. Thus so is the inverse limit of these rings, which is the I -adic completion $\widehat{(R_P)_I}$ of R_P . Since $I \subset \mathfrak{m}$, the I -adic completion of R_P is contained in the \mathfrak{m} -adic completion \widehat{R}_P of R_P . But \widehat{R}_P is the smallest \mathfrak{m} -adically complete ring containing R_P ; so the containment $\widehat{(R_P)_I} \subseteq \widehat{R}_P$ is an equality. This proves the claim and hence the result. \square

Proposition 7.5. *Let F be a function field in one variable over a complete discretely valued field K with valuation ring T . Let \widehat{X} be a regular T -curve with function field F . If P is a closed point of \widehat{X} and v is a discrete valuation on F_P then the restriction of v to F is a discrete valuation on F .*

Proof. To prove the assertion we need to show that the restriction of v to F is non-trivial; i.e. $v(f) \neq 0$ for some $f \in F^\times$.

By Proposition 3.3, there is a finite morphism $f : \widehat{X} \rightarrow \mathbb{P}_T^1$ taking P to the point P' at infinity on the closed fiber \mathbb{P}_k^1 of \mathbb{P}_T^1 . Thus F is a finite extension of the function field $F' = K(x)$ of \mathbb{P}_T^1 . Let v' be the restriction of v to $F'_{P'}$, the fraction field of the complete local ring of \mathbb{P}_T^1 at P' . Now F_P is a finite field extension of $F'_{P'}$; so [Bou72, Proposition VI.8.1.1] implies that the value group of v' has finite index in that of v . Hence v' is a (non-trivial) discrete valuation on $F'_{P'}$.

The ring \widehat{R}_P is contained in the valuation ring $R_v \subset F_P$ of v by Proposition 7.1, taking $R = \widehat{R}_P$ there. Thus $\widehat{R}'_{P'} = \widehat{R}_P \cap F'_{P'}$ is contained in $R_{v'} := R_v \cap F'_{P'}$, the valuation ring of v' . Hence $v'(u) = 0$ for every unit $u \in \widehat{R}'_{P'}$.

Let $a \in R_{v'} \subset F'_{P'}$ be a uniformizer for v' and let η' be the generic point of \mathbb{P}_k^1 . Thus $v'(a) > 0$. By [HH10, Corollary 5.6] (taking $\widehat{R} = \widehat{R}'_{\{P'\}}$, $\widehat{R}_1 = \widehat{R}'_{P'}$, $R_2 = \widehat{R}'_{\eta'}$, and $\widehat{R}_0 = \widehat{R}'_{\wp'}$), and writing $a \in F'_{P'}$ as a ratio of elements in $\widehat{R}'_{P'}$, there exist $u_1 \in \widehat{R}'_{P'}$ and $a_1 \in F'_{\{P'\}}$ such that $a = u_1 a_1$. By [HH10, Corollary 4.8], there exist $u_2 \in \widehat{R}'_{\{P'\}}^\times$ and $b \in F'$ such that $a_1 = u_2 b$. Thus $a = u b$ with $u = u_1 u_2 \in \widehat{R}'_{P'}^\times$. But $v'(u) = 0$; so $v'(b) > 0$. Thus the restriction of v' to F' is non-trivial. Hence so is the restriction of v to F . \square

Proposition 7.6. *Let F be a function field in one variable over a complete discretely valued field K with valuation ring T . Let \widehat{X} be a regular T -curve with function field F , and let E/F be a finite field extension. Then the following are equivalent:*

- (i) E/F is a split extension.
- (ii) There is a split cover \widehat{Y} of \widehat{X} with function field E .

Proof. If E/F is a split extension, let \widehat{Y} be the normalization of \widehat{X} in E . For every codimension one point Q on \widehat{X} , the fraction field F_Q of the complete local ring \widehat{R}_Q at Q is the completion of F at the discrete valuation v defined by Q . By definition of split extensions, $E \otimes_F F_v$ is a product of copies of F_v . That is, E/F is split over F_Q . Since \widehat{X} is regular, Corollary 5.3 asserts that $\widehat{Y} \rightarrow \widehat{X}$ is a split cover.

Conversely, if $\widehat{Y} \rightarrow \widehat{X}$ is a split cover with function field E , let v be a discrete valuation on F . By Lemma 7.3 above, v has a center Q on \widehat{X} that is not the generic point, and $F_Q \subseteq F_v$. By definition of a split cover, $\widehat{Y} \rightarrow \widehat{X}$ splits over the point Q (since it is not the generic point of \widehat{X}); and so by Hensel's Lemma it splits over \widehat{R}_Q and thus over F_Q . The containment $F_Q \subseteq F_v$ implies that $\widehat{Y} \rightarrow \widehat{X}$ splits over F_v , and hence so does E/F . \square

Remark 7.7. The proof of Proposition 7.6 shows that condition (ii) follows from an a priori weaker form of condition (i), viz. it suffices to assume that E/F is split over each discrete valuation of F that is given by a codimension one point on the chosen regular model \widehat{X} . Since (ii) implies (i), it follows that an extension E/F is split if and only if it is split over F_v for each v in that smaller set of valuations.

Since split covers are étale, Proposition 7.6 implies that every finite split extension of F is separable; and hence so is F^{split}/F . Also, F^{split} is invariant under the absolute Galois group of F . Hence the above proposition and Corollary 6.4 yield the following:

Corollary 7.8. *Let F be a one-variable function field over the fraction field of a complete discrete valuation ring. Then F^{split}/F is a Galois extension; and for any regular model \widehat{X} of F , there is a natural identification*

$$\text{Gal}(F^{\text{split}}/F) = \pi_1^{\text{split}}(\widehat{X})$$

of free profinite groups on finitely many generators. Hence $\pi_1^{\text{split}}(\widehat{X})$ and the fundamental group of the reduction graph are independent of the choice of the regular model \widehat{X} for F .

Combining this with Corollary 6.5 if G is rational, we obtain that $\text{III}_0(\widehat{X}, G)$ does not depend on the regular model \widehat{X} of F (as F^{split} does not). We may then write

$$\text{III}_0(F, G) := \text{III}_0(\widehat{X}, G),$$

where \widehat{X} is any regular model of F .

Lemma 7.9. *Let \widehat{X} be a regular projective curve over a complete discrete valuation ring T , with closed fiber X .*

- (a) *For every point P of X , there is an inclusion $F^{\text{split}} \hookrightarrow F_P$ of F -algebras.*
- (b) *For any irreducible component $X_0 \subseteq X$, and any non-empty affine open subset $U \subset X_0$ that does not meet any other irreducible component of X , there is an inclusion $F^{\text{split}} \hookrightarrow F_U$ of F -algebras.*

Proof. (a) Let \bar{F} be an algebraic closure of F . The field $F^{\text{split}} \subset \bar{F}$ is a union of finite split field extensions F_i/F , each of which is the function field of a finite split cover $\widehat{X}_i \rightarrow \widehat{X}$ (by Proposition 7.6), say of degree n_i . Since this cover is split, it is also split over F_P by Proposition 5.1, and thus the pullback of the cover to $\text{Spec}(F_P)$ is a disjoint union of finitely many copies of $\text{Spec}(F_P)$. Equivalently, $F_i \otimes_F F_P$ is F_P -isomorphic to a direct product of n_i copies of F_P . Composing with a projection map to F_P shows that the set S_i of F -homomorphisms $F_i \hookrightarrow F_P$ is non-empty; here S_i is finite since F_i is finite over F . Thus the sets S_i form an inverse system of non-empty finite sets, and so the inverse limit S is non-empty. Pick an element of S . This defines a compatible system of inclusions $F_i \hookrightarrow F_P$ as i varies, and so defines an inclusion $F^{\text{split}} \hookrightarrow F_P$.

(b) Let \mathcal{P} be the finite subset of X that consists of the complement of U in X_0 together with all the points of X at which distinct irreducible components of X meet. Let \mathcal{U} be the set of connected components of the complement of \mathcal{P} in X . Thus $U \in \mathcal{U}$. The proof is now the same as that of part (a), except that Corollary 5.5 replaces Proposition 5.1. \square

Theorem 7.10. *Let G be a rational linear algebraic group defined over F . Then*

$$\text{III}_0(F, G) = H^1(F^{\text{split}}/F, G).$$

Proof. Let $\iota : H^1(F^{\text{split}}/F, G) \rightarrow H^1(F, G)$ be the natural inclusion. By Lemma 7.9(a), F^{split} includes into F_P for each point P of X . Hence the image of ι is in $\text{III}_0(F, G)$. It suffices to show that ι surjects onto $\text{III}_0(F, G)$.

Write $\bar{G} = G/G^0$ and let \hat{X} be a regular model of F over T . To show surjectivity, consider an element $\alpha \in \text{III}_0(F, G)$, with image $\bar{\alpha} \in \text{III}_0(F, \bar{G})$. By Corollary 5.6, $\bar{\alpha}$ corresponds to a \bar{G} -Galois split cover $\hat{Y} \rightarrow \hat{X}$, with function field extension E/F . Write α_E for the image of α under $\text{III}_0(F, G) \rightarrow \text{III}_0(E, G)$.

Since E/F is Galois, $E \otimes_F E$ is a direct product of copies of E , and hence the induced element $(\bar{\alpha})_E \in \text{III}_0(E, \bar{G}) \subseteq H^1(E, \bar{G})$ is trivial. But this is the image of α_E under $\text{III}_0(E, G) \rightarrow \text{III}_0(E, \bar{G})$, and this map is bijective by Theorem 5.10. So α_E is trivial. Since $E \subseteq F^{\text{split}}$, it follows that α is in fact in $H^1(F^{\text{split}}/F, G)$, as desired. \square

8 Local-global principles with respect to valuations

In this section, we consider a local-global map with respect to the discrete valuations on a field F ; this was previously considered in [COP02] and [CGP04]. Let Ω_F denote the set of equivalence classes of discrete valuations v on F (i.e., valuations with value group isomorphic to \mathbb{Z}).

Given an algebraic group G over F , we define

$$\text{III}(F, G) := \ker(H^1(F, G) \rightarrow \prod_{v \in \Omega_F} H^1(F_v, G)).$$

Thus $\text{III}(F, G)$ classifies G -torsors over F that become trivial over each F_v . Again, this is a pointed set, and is a group if G is commutative.

We will study $\text{III}(F, G)$ in the situation we have been considering; i.e. F is a one-variable function field over a complete discretely valued field K , and G is a linear algebraic group over F . As before, the valuation ring of K will be denoted by T , and we will consider regular models of F , i.e., regular projective T -curves \hat{X} with function field F . In particular, we consider the relationship between $\text{III}(F, G)$ and $\text{III}_0(\hat{X}, G)$, using this to obtain information about local-global principles for torsors in terms of discrete valuations. (For now, we do *not* assume that G is rational, and so we cannot conclude from Section 7 that $\text{III}_0(\hat{X}, G)$ depends only on the function field F .) Theorem 8.7 gives a criterion for the equality of $\text{III}(F, G)$ and $\text{III}_0(\hat{X}, G)$, and this is then used to obtain specific conditions guaranteeing this equality (Theorem 8.10). In particular, Corollary 8.11 gives a criterion under which $\text{III}(F, G)$ is finite and there is a necessary and sufficient condition for it to vanish.

Remark 8.1. The above definition of $\text{III}(F, G)$ was used in [CPS08], and it was also considered (without this notation) in [COP02]. In [CGP04] and [BKG04], a slightly different version of $\text{III}(F, G)$ was considered. There, one takes only those discrete valuations that arise from codimension one points on blow-ups of a given regular model. In our situation, these two sets of discrete valuations are the same if the residue field k of T is algebraically closed; but more generally the sets are unequal. In that case, our $\text{III}(F, G)$ is contained in

the variant version, and it would be interesting to know if this latter containment is always an equality. As the proofs below show, the results stated here about $\text{III}(F, G)$ each hold with either version, and some of the results hold even if one just considers the discrete valuations associated to codimension one points on a fixed regular model (as was the case with Proposition 7.6/Remark 7.7).

Proposition 8.2. *Let F be a one-variable function field over a complete discretely valued field K , and let G be a linear algebraic group over F . Let \widehat{X} be a regular model of F . Then $\text{III}_0(\widehat{X}, G) \subseteq \text{III}(F, G)$ as subsets of $H^1(F, G)$.*

Proof. Let $\alpha \in H^1(F, G)$ be an element of $\text{III}_0(\widehat{X}, G)$. Then α has trivial image in $H^1(F_P, G)$ for every $P \in X$. Let $v \in \Omega_F$. By Proposition 7.4, there is some point P on the closed fiber X of \widehat{X} such that F_P is contained in F_v . Hence α has trivial image in $H^1(F_v, G)$. Thus α is in $\text{III}(F, G)$. \square

Proposition 8.3. *Let \widehat{X} be a regular projective curve over a complete discrete valuation ring T , with function field F and closed fiber X . Let $X^{(0)}$ be the set of closed points of X . For any linear algebraic group G over F , the image of $\text{III}(F, G)$ under the natural map*

$$\phi : H^1(F, G) \rightarrow \prod_{P \in X^{(0)}} H^1(F_P, G)$$

is $\prod'_{P \in X^{(0)}} \text{III}(F_P, G)$, the subset of the product in which all but finitely many entries are trivial.

Proof. For any $P \in X^{(0)}$ and any $v \in \Omega_{F_P}$, the restriction v_0 of v to F is a discrete valuation in Ω_F , by Proposition 7.5. If α is in $\text{III}(F, G)$ then the image of α in $H^1(F_{v_0}, G)$ is trivial; and hence so is its image in $H^1((F_P)_v, G)$. But this is the same as the image in $H^1((F_P)_v, G)$ of the P -component of $\phi(\alpha)$. Thus $\phi(\alpha)$ and hence $\phi(\text{III}(F, G))$ are contained in $\prod_{P \in X^{(0)}} \text{III}(F_P, G)$.

Let X_1, \dots, X_n be the irreducible components of X , and let η_i be the generic point of X_i . Then η_i is a codimension one point of \widehat{X} , and so corresponds to a discrete valuation in Ω_F . Thus if α is an element of $\text{III}(F, G)$, then the image of α is trivial in $H^1(F_{\eta_i}, G)$. That is, α corresponds to a G -torsor over F that has an F_{η_i} -point. By Proposition 5.8, it also has an F_{U_i} -point, for some affine open subset $U_i \subset X_i$ that does not meet any other component. Thus the image of α is trivial in $H^1(F_{U_i}, G)$, and hence also trivial in $H^1(F_P, G)$ for all $P \in U_i$. Since there are only finitely many points of X that do not lie in any of the sets U_i , it follows that $\phi(\text{III}(F, G))$ is indeed contained in $\prod'_{P \in X^{(0)}} \text{III}(F_P, G)$.

To prove the result we will show that $\text{III}(F, G)$ surjects onto $\prod'_{P \in X^{(0)}} \text{III}(F_P, G)$ under ϕ . So consider a finite set S of closed points of X , and elements $\alpha_P \in \text{III}(F_P, G)$ for $P \in S$. For each $P \in X^{(0)}$ that is in the complement of S , let α_P be the trivial element of $\text{III}(F_P, G)$. We wish to find an element $\alpha \in \text{III}(F, G)$ whose image in $\text{III}(F_P, G)$ is α_P for all $P \in X^{(0)}$.

After enlarging S , we may assume that S contains a point on each irreducible component X_i of X , and includes all the points at which X is not unbranched. Since α_P is in $\text{III}(F_P, G) \subseteq H^1(F_P, G)$, the restriction of α_P to $H^1(F_\wp, G)$ is trivial for any height one prime \wp of \widehat{R}_P . In particular, this holds if \wp is a branch of X at P .

Let U_i now be the complement of $S \cap X_i$ in X_i , and let α_{U_i} be the trivial element of $H^1(F_{U_i}, G)$. Thus α_{U_i} is an element of $\text{III}(F_{U_i}, G)$, and its restriction to $H^1(F_\varphi, G)$ is trivial for any branch φ at a point $P \in S \cap X_i$. Thus α_{U_i}, α_P induce isomorphic (trivial) elements of $H^1(F_\varphi, G)$, if $P \in S \cap X_i$ and φ is a branch at P on the closure of U_i . Hence Theorem 3.5 applies; and so there is some $\alpha \in H^1(F, G)$ that induces $\alpha_P \in H^1(F_P, G)$ for each $P \in S$ and induces $\alpha_{U_i} \in H^1(F_{U_i}, G)$ for each i . Since α_{U_i} is trivial, it follows that α induces the trivial element of $H^1(F_P, G)$ for each P in the complement of S .

It remains to show that α lies in $\text{III}(F, G)$. So consider any $v \in \Omega_F$. By Proposition 7.4, there is a point $P \in X$ such that F_P is contained in F_v . If P is a codimension one point of \hat{X} , then $P = \eta_i$ for some i ; and then the image of α in $H^1(F_{\eta_i}, G)$ is trivial since its image in $H^1(F_{U_i}, G)$ is trivial and since F_{η_i} contains F_{U_i} . The other possibility is that P is a closed point of X . Then the discrete valuation on F_v restricts to a discrete valuation v_P on F_P that in turn restricts to v on F ; and the completion of F_P at v_P is just F_v . Since α_P lies in $\text{III}(F_P, G)$, the image of α_P in $H^1(F_v, G)$ is trivial. But α_P is the image of α in $H^1(F_P, G)$. Hence the image of α in $H^1(F_v, G)$ is also trivial. Thus α lies in $\text{III}(F, G)$. \square

Proposition 8.4. *Let \hat{X} be a regular projective curve over a complete discrete valuation ring T , with function field F and closed fiber X . Let $X^{(0)}$ be the set of closed points of X . For any linear algebraic group G over F , there is an exact sequence*

$$1 \rightarrow \text{III}_0(\hat{X}, G) \xrightarrow{\iota} \text{III}(F, G) \xrightarrow{\phi} \prod'_{P \in X^{(0)}} \text{III}(F_P, G) \rightarrow 1 \quad (*)$$

of pointed sets, in which the map ι is an inclusion.

Proof. By Proposition 8.2, there is an inclusion $\iota : \text{III}_0(\hat{X}, G) \rightarrow \text{III}(F, G)$, given by viewing each as a subset of $H^1(F, G)$. By Proposition 8.3, there is a surjection $\phi : \text{III}(F, G) \rightarrow \prod'_{P \in X^{(0)}} \text{III}(F_P, G)$, whose entries are the natural maps to $\text{III}(F_P, G) \subseteq H^1(F_P, G)$. The composition $\phi\iota$ is trivial, since $\text{III}_0(\hat{X}, G)$ is the kernel of the map $H^1(F, G) \rightarrow \prod_{P \in X} H^1(F_P, G)$. Also, any element of $\text{III}(F, G)$ has trivial image in $H^1(F_P, G)$ for any codimension one point $P \in \hat{X}$, and in particular for the generic point of any irreducible component of X . Hence any element in the kernel of ϕ is also in the kernel of $H^1(F, G) \rightarrow \prod_{P \in X} H^1(F_P, G)$; i.e. is in $\text{III}_0(\hat{X}, G)$. So the sequence is exact. \square

Proposition 8.4 shows that $\text{III}_0(\hat{X}, G)$ is the kernel of ϕ . But since $(*)$ is in general just an exact sequence of pointed sets, describing the kernel does not describe the fibers of ϕ . But as in Corollary 2.5, we can describe those fibers by using the bijection between $H^1(F, G)$ and $H^1(F, G^\tau)$ that takes the class $[\tau] \in H^1(F, G)$ of $\tau \in Z^1(F, G)$ to the neutral element. Namely, the fiber of ϕ containing $[\tau] \in \text{III}(F, G)$ is carried to the kernel of the corresponding map for the twisted group G^τ ; and this is just $\text{III}_0(\hat{X}, G^\tau)$, by Proposition 8.4 applied to G^τ . We thus obtain the following corollary:

Corollary 8.5. *In the situation of Proposition 8.4, let $\tau \in Z^1(F, G)$ be a cocycle whose class $[\tau]$ lies in $\text{III}(F, G)$. Then the fiber of $\phi : \text{III}(F, G) \rightarrow \prod'_{P \in X^{(0)}} \text{III}(F_P, G)$ that contains $[\tau]$ is in natural bijection with $\text{III}_0(\hat{X}, G^\tau)$ as pointed sets.*

By Proposition 8.4, if $\text{III}(F_P, G)$ is trivial for every closed point P on the closed fiber X of some regular model \widehat{X} of F , then $\text{III}(F, G) = \text{III}_0(\widehat{X}, G)$. A related result appears at Theorem 8.7 below. First we prove a lemma.

Lemma 8.6. *Let E be the fraction field of a two-dimensional complete regular local ring A , and let G be a finite linear algebraic group over E . Then $\text{III}(E, G)$ is trivial.*

Proof. Let $\alpha \in \text{III}(E, G) \subseteq H^1(E, G)$. Then α defines an étale E -algebra; let B be the integral closure of A in this algebra. Thus the morphism $\text{Spec}(B) \rightarrow \text{Spec}(A)$ is finite and generically separable. Moreover it is unramified over the codimension one points of $\text{Spec}(A)$, since $\alpha \in \text{III}(E, G)$. Since A is regular and B is normal, it follows by Purity of Branch Locus that $\text{Spec}(B) \rightarrow \text{Spec}(A)$ is unramified, and hence is an étale cover.

Let $\{x, y\}$ be a system of parameters for $\text{Spec}(A)$ at its closed point P . By hypothesis, the torsor defined by α becomes trivial over the x -adic completion E_x of E . Thus there is a section of the étale cover $\text{Spec}(B) \rightarrow \text{Spec}(A)$ over $\text{Spec}(A_x)$, where A_x is the completion of the local ring of A at the height one prime (x) . Let $V \subset \text{Spec}(A)$ be the closed subset defined by (x) . The étale cover $\text{Spec}(B) \rightarrow \text{Spec}(A)$ thus has a section over the generic point (x) of V , and hence has a section over all of V . Hence it has trivial residue field extension at a point lying over the closed point P of V . Since $\text{Spec}(B) \rightarrow \text{Spec}(A)$ is étale, Hensel's Lemma implies that $\text{Spec}(B) \rightarrow \text{Spec}(A)$ has a section. Thus the torsor defined by α has a section over $\text{Spec}(E)$; i.e. α corresponds to the trivial G -torsor over E . This shows that $\text{III}(E, G)$ is trivial. \square

If G is a rational linear algebraic group over F , then $G(E) \rightarrow (G/G^0)(E)$ is surjective for every E/F . Hence $H^1(E, G^0) \rightarrow H^1(E, G)$ is injective by the cohomology exact sequence [Ser73, I, 5.5, Proposition 38], and thus $\text{III}(F_P, G^0) \subseteq \text{III}(F_P, G)$ for every closed point $P \in X$. For such groups, the vanishing of the a priori smaller set $\text{III}(F_P, G^0)$ suffices to obtain the equality stated after Corollary 8.5, as the following result shows:

Theorem 8.7. *Let \widehat{X} be a regular projective curve over a complete discrete valuation ring T , with function field F and closed fiber X . Let G be a linear algebraic group over F and assume that for each $P \in X$ and each valuation v on F_P , the homomorphism $G((F_P)_v) \rightarrow (G/G^0)((F_P)_v)$ is surjective. Suppose moreover that $\text{III}(F_P, G^0)$ is trivial for every closed point P of \widehat{X} . Then $\text{III}(F, G) = \text{III}_0(\widehat{X}, G)$ as subsets of $H^1(F, G)$.*

Proof. Let \widehat{X} be a regular model of F over T . Let $X^{(0)}$ denote the set of closed points of its closed fiber X . By Proposition 8.4, it suffices to show that $\text{III}(F_P, G)$ is trivial for each closed point $P \in X^{(0)}$.

Let $P \in X^{(0)}$. By [Ser73, I, 5.5, Proposition 38], the sequence of pointed sets

$$H^1(F_P, G^0) \rightarrow H^1(F_P, G) \rightarrow H^1(F_P, \bar{G})$$

is exact, where $\bar{G} = G/G^0$.

Now let $\alpha \in \text{III}(F_P, G) \subseteq H^1(F_P, G)$. Then the image of α in $H^1(F_P, \bar{G})$ lies in $\text{III}(F_P, \bar{G})$ and hence is trivial by Lemma 8.6. Thus α is the image of an element of $H^1(F_P, G^0)$. By

the surjectivity assumption this element necessarily lies in $\text{III}(F_P, G^0)$ (see the arguments in the first and last paragraphs of the proof of Corollary 2.6). But $\text{III}(F_P, G^0)$ is trivial, and so α is trivial. This shows that $\text{III}(F_P, G)$ is trivial and completes the proof. \square

Results about the vanishing of $\text{III}(E, G)$ for algebraic groups G over fraction fields E of regular complete local rings give applications of the above theorem. In particular, there is the next result, following a suggestion of J.-L. Colliot-Thélène, and using a strategy that is similar to that used in the proof of Lemma 8.6. Here, as in [DeGr70], Exp. XIX, Définition 2.7, we say that a group scheme G over a base scheme S is *reductive* if it is affine and smooth over S , and its geometric fibers are connected and reductive (meaning they have trivial unipotent radical).

Lemma 8.8. *Let E be the fraction field of a two-dimensional complete regular local ring A , and let G be a reductive group scheme over A . Then $\text{III}(E, G)$ is trivial.*

Proof. If $\alpha \in \text{III}(E, G) \subseteq H^1(E, G)$, then the corresponding G -torsor over $\text{Spec}(E)$ is unramified at the codimension one points of $\text{Spec}(A)$. It therefore follows that α is induced by an element α_A of $H^1(\text{Spec}(A), G)$, as shown in [CPS08, Theorem 4.2(i)]. (That result considered not our space $\text{Spec}(A)$ but rather a two-dimensional regular scheme that is projective over a complete discrete valuation ring; but the only hypothesis that was used in the proof was that the scheme is two-dimensional and regular with function field E .)

Let $\{x, y\}$ be a system of parameters for $\text{Spec}(A)$ at its closed point P . By hypothesis, the torsor defined by α_A becomes trivial over the x -adic completion E_x of E . According to [Nis84], since G is reductive, the natural map $H^1(A_x, G) \rightarrow H^1(E_x, G)$ has trivial kernel, where the discrete valuation ring A_x is the completion of the local ring of A at the height one prime (x) . Thus the image of α_A in $H^1(A_x, G)$ is trivial. Hence so is its image in $H^1(A_x/xA_x, G)$. Since A/xA is a discrete valuation ring with fraction field A_x/xA_x , it follows from [Nis84] that the image of α_A in $H^1(A/xA, G)$ is trivial. Hence so is its image in $H^1(A/(x, y), G)$. Thus the G -torsor over A given by α_A has a rational point over the residue field $A/(x, y)$ at P . But A is complete. So this point lifts to an A -point on the torsor. Thus this torsor is trivial, and hence so is the G -torsor over E defined by α . This shows that $\text{III}(E, G)$ is trivial. \square

Remark 8.9. The assertion cited from [Nis84] was stated for reductive groups, but the proof was given there only for semi-simple groups. The proof in the general reductive case was given in [Gil94, Théorème I.1.2.2].

Theorem 8.10. *Let F be a one-variable function field over the fraction field of a complete discrete valuation ring T , and let \widehat{X} be a regular model for F , with closed fiber X . Let G be a linear algebraic group over F such that for each $P \in X$ and each valuation v on F_P , the homomorphism $G((F_P)_v) \rightarrow (G/G^0)((F_P)_v)$ is surjective. Then $\text{III}_0(\widehat{X}, G)$ equals $\text{III}(F, G)$ in each of the following situations:*

- (i) G^0 is a rational variety and the residue field of T is algebraically closed of characteristic zero; or

- (ii) G^0 is the generic fiber of a reductive group scheme over a regular model \widehat{X} of F ; or
- (iii) G^0 is semi-simple and simply connected, and the residue field of T is algebraically closed of characteristic zero.

In particular, in any of these cases, $\text{III}_0(\widehat{X}, G)$ is independent of the choice of the regular model \widehat{X} of F .

Proof. By Theorem 8.7, it suffices to show that $\text{III}(F_P, G^0)$ is trivial for every closed point P of \widehat{X} . In case (i) this follows from [BKG04, Corollary 7.7]; and in case (ii) it follows from Lemma 8.8 above, via base change from \widehat{X} to $\text{Spec}(\widehat{R}_P)$. In case (iii) it follows from the fact that $H^1(F_P, G^0)$, which contains $\text{III}(F_P, G^0)$, is trivial by [COP02, Theorem 5.1]. \square

In particular, if G is rational over F and the residue field of T is algebraically closed of characteristic zero, then the above conclusions about $\text{III}_0(\widehat{X}, G)$ hold. In this case, the last assertion of the theorem was previously observed (see the comment after Corollary 7.8).

Combining Theorem 8.10 with Corollary 6.5 then gives:

Corollary 8.11. *Let G be a rational linear algebraic group over F . If the residue field of T is algebraically closed of characteristic zero, or if condition (ii) of Theorem 8.10 holds, then $\text{III}(F, G)$ is finite. Under either of these hypotheses, $\text{III}(F, G)$ is trivial if and only if either G is connected or $F^{\text{split}} = F$ (or, equivalently, if the reduction graph of some regular model of F is a tree).*

It would be interesting to know if the conclusion of Theorem 8.10 always holds even without assuming any of the three additional hypotheses. If so, then in particular the conclusion of Corollary 8.11 would hold for rational linear algebraic groups even without the assumption on the residue field of T .

9 Applications

In this section we apply the previous results in order to obtain local-global principles for quadratic forms and central simple algebras. We begin by considering homogeneous spaces which are not necessarily torsors.

9.1 Applications to homogeneous spaces

Until now our focus has been on local-global principles for torsors, i.e. for principal homogeneous spaces. Some results also carry over to other homogeneous spaces. As in [HHK09], if H is an F -variety on which a linear algebraic group acts, then we say that G *acts transitively on the points of H* if $G(E)$ acts transitively on $H(E)$ for every field extension E of F . As observed in Corollary 2.8, if a local-global principle holds for G -torsors then it holds for all homogeneous G -spaces that satisfy the above transitivity assumption. There, the context was for factorization inverse systems. But it holds in particular for the situation of patches,

i.e. that of Notation 3.2, since by Corollary 3.4 they form such a system satisfying the hypotheses of Corollary 2.8. Using Proposition 5.8, the corresponding property holds in the context of points on the closed fiber, in parallel to [HHK09], Theorem 3.7.

Theorem 9.1. *Let F be a one variable function field over a complete discretely valued field with valuation ring T , and let \widehat{X} be a normal model for F over T . If G is a linear algebraic group over F such that $\text{III}_0(\widehat{X}, G)$ is trivial, then there is a local-global principle with respect to points on the closed fiber X for all F -varieties H for which G acts transitively on the points. That is, if $H(F_P)$ is non-empty for every $P \in X$, then $H(F)$ is non-empty. In particular, this local-global principle holds if G is connected and rational.*

Proof. Let X_1, \dots, X_r be the irreducible components of X , and let η_i be the generic point of X_i . Since $H(F_{\eta_i})$ is non-empty, so is $H(F_{U_i})$ for some affine dense open subset U_i of X_i that does not meet any other X_j , by Proposition 5.8. Let \mathcal{U} be the collection of the sets U_i , and let \mathcal{P} be the complement of their union. As was pointed out at the beginning of Section 5, $\text{III}_{\mathcal{P}}(\widehat{X}, G)$ is a subset of $\text{III}_0(\widehat{X}, G)$ and thus trivial by assumption. Hence by Corollary 2.8 applied to the factorization inverse system given by the patches, if $H(F_{\xi}) \neq \emptyset$ for each $\xi \in \mathcal{P} \cup \mathcal{U}$, then $H(F) \neq \emptyset$. The last assertion now follows from Corollary 6.5. \square

It would be desirable to prove an analogous assertion in the context of discrete valuations, viz. that if $\text{III}(F, G)$ is trivial then for any G -space H with transitive action, there is an F -point on H provided that there is an F_v -point for each discrete valuation v . In this general direction we have the following result:

Theorem 9.2. *Let T be a complete discrete valuation ring with fraction field K , such that its residue field k is algebraically closed of characteristic zero. Let F be a one-variable function field over K with a regular model \widehat{X} , and let G be a linear algebraic group over F such that $\text{III}_0(\widehat{X}, G)$ is trivial. If H is a smooth projective F -variety with a G -action that is transitive on points, and if $H(F_v)$ is non-empty for every discrete valuation v on F , then $H(F)$ is non-empty.*

Proof. As usual, let X denote the closed fiber of \widehat{X} . By Proposition 7.5, for every closed point $P \in X$ and discrete valuation v on F_P , the restriction of v to F is a discrete valuation v_0 . Hence for every such P , $H((F_P)_{v_0})$ is non-empty, as it contains $H(F_{v_0})$. By Corollary 5.7 of [CGP04], $H(F_P)$ is non-empty.

Next, consider a generic point η of an irreducible component X_0 of X . Then $H(F_{\eta})$ is non-empty since $F_{\eta} = F_v$, where v is the discrete valuation on F corresponding to the codimension one point $\eta \in \widehat{X}$.

Thus $H(F_P)$ is non-empty for every point $P \in X$. Since $\text{III}_0(\widehat{X}, G)$ is trivial, it follows from Theorem 9.1 that $H(F)$ is non-empty. \square

Note that in the above two results, the hypothesis on G is satisfied if $\text{III}(F, G)$ is trivial, since $\text{III}_0(\widehat{X}, G)$ is contained in $\text{III}(F, G)$ by Proposition 8.4. Also, this hypothesis holds if G is a rational linear algebraic group over F such that either G is connected or the reduction graph associated to F is a tree, by Corollary 6.5.

9.2 Applications to quadratic forms

Below we prove local-global principles for quadratic forms in terms of points on the closed fiber and in terms of valuations. These results concern whether a quadratic form is isotropic or hyperbolic; the value of the Witt index of a form; and the Witt group of a field.

As before, let T be a complete discrete valuation ring with uniformizer t , fraction field K , and residue field k , and let F be a one-variable function field over K . In this section, we assume that K does not have characteristic 2. Let \widehat{X} be a normal model of F over T .

In the situation of Notation 3.2, a local-global principle for isotropy was shown in [HHK09, Theorem 4.2] for quadratic forms over F of dimension unequal to two, in terms of the fields F_P and F_U . An analogous local-global principle in terms of completions F_v at valuations was shown in [CPS08, Theorem 3.1], using the result in [HHK09] and a theorem of Springer ([Lam05, Proposition VI.1.9(2)]); this is an analog of the classical Hasse-Minkowski theorem. The next result proves the analog in terms of points on the closed fiber X of \widehat{X} . (We have learned that this result also appears in the Ph.D. thesis of David Grimm. It also follows from [CPS08, Theorem 3.1] together with Proposition 7.4.)

Theorem 9.3. *If q is a quadratic form over F of dimension unequal to two, and if q is isotropic over F_P for every $P \in X$, then q is isotropic over F .*

Proof. Without loss of generality, we may assume that q is regular of dimension $n \geq 3$. Then $O(q)$ and its identity component $SO(q)$ act transitively on the points of the projective quadric hypersurface H defined by q (for details, see the proof of Theorem 4.2 of [HHK09]). By Remark 4.1 of that article, the group $SO(q)$ is rational. Thus by Theorem 9.1, $H(F)$ is non-empty provided that each $H(F_P)$ is; this is equivalent to the desired assertion. \square

Recall that two quadratic forms q, q' are *Witt equivalent* if $q \perp h \cong q' \perp h'$, where h, h' are hyperbolic forms. By the Witt decomposition ([Lam05, Theorem I.4.1]), every regular form q is Witt equivalent to an anisotropic form; i.e. $q = q_a \perp q_h$ with q_a anisotropic and q_h hyperbolic. Here the *Witt index* of q is $i_W(q) = \frac{1}{2} \dim q_h$.

Corollary 9.4. *If q is a regular quadratic form then*

$$i_W(q) = \min_{P \in X} i_W(q_{F_P}) \tag{*}$$

unless q is Witt equivalent to an anisotropic binary form that becomes isotropic over each F_P . In that exceptional case,

$$i_W(q) = \min_{P \in X} i_W(q_{F_P}) - 1. \tag{**}$$

Proof. Write $q = q_a \perp q_h$, as above. If q_a is not binary, then since it is anisotropic over F , it is also anisotropic over some F_P , by Theorem 9.3. The equality (*) now follows.

If q_a is binary and some $(q_a)_P$ is anisotropic, then $i_W(q_a) = \min_{P \in X} i_W((q_a)_{F_P}) = 0$. Thus (*) holds in this case. In the remaining (exceptional) case, $i_W(q_a) = 0$ and $\min_{P \in X} i_W((q_a)_{F_P}) = 1$, hence (**) holds. \square

To any rational linear algebraic group G over F , we may associate the pointed set $\text{III}_0(F, G)$, which is equal to $\text{III}_0(\widehat{X}, G)$ for any regular model \widehat{X} of F over T . Recall that under Hypothesis 5.4, the fundamental group of the reduction graph of a regular model of F also depends only on F , by Corollary 7.8. As a result, we may simply write $\pi_1(\Gamma)$ for this common fundamental group, where Γ is the reduction graph of any regular model of F .

Theorem 9.5. *Let F be a one variable function field over a complete discretely valued field K of characteristic not equal to 2, and let \widehat{X} be a regular model of F . Then the kernel of the natural homomorphism of Witt groups $\pi : W(F) \rightarrow \prod_{P \in X} W(F_P)$ is isomorphic to $\text{III}_0(F, \mathbb{Z}/2\mathbb{Z})$. Moreover, both groups are isomorphic to the elementary abelian two-group $\text{Hom}(\pi_1(\Gamma), \mathbb{Z}/2\mathbb{Z})$, and each element in the kernel is represented by a quadratic form of dimension two.*

Proof. Let h denote a hyperbolic plane. Then $\text{SO}(h)$ is rational by [HHK09], Remark 4.1; and hence so is $\text{O}(h)$, since each of the two components has a rational point. Now $H^1(F, \text{O}(h))$ classifies the equivalence classes of regular two-dimensional quadratic forms over F , with the distinguished element corresponding to the quadratic form h (see [KMRT98, Proposition VII.29.1 and VII.29.28]). Each element in $\ker \pi \subseteq W(F)$ is represented by an anisotropic quadratic form that becomes hyperbolic over each F_P . Since a non-trivial hyperbolic form is isotropic, it follows from Theorem 9.3 that such a form is binary. Thus there is a natural bijection of pointed sets $\ker \pi \rightarrow \text{III}_0(F, \text{O}(h))$. Since $\text{O}(h)$ is rational, and its quotient by its identity component is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, there is a bijection of $\text{III}_0(F, \text{O}(h))$ to $\text{III}_0(F, \mathbb{Z}/2\mathbb{Z})$, by Corollary 6.5. We claim that the composition $\ker \pi \rightarrow \text{III}_0(F, \mathbb{Z}/2\mathbb{Z})$ is a homomorphism, and hence an isomorphism. It suffices to show that the composition $\ker \pi \rightarrow \text{III}_0(F, \mathbb{Z}/2\mathbb{Z}) \subseteq H^1(F, \mathbb{Z}/2\mathbb{Z}) = F^\times / (F^\times)^2$ is. But this composition takes the diagonal quadratic form $\langle 1, -a \rangle$ to the square class of a , for $a \in F^\times$. Since $\langle 1, -a \rangle \perp \langle 1, -b \rangle$ is Witt equivalent to $\langle 1, -ab \rangle$, the claim follows.

The remaining assertion now follows directly from Corollary 5.6. \square

Thus, the local-global map $\pi : W(F) \rightarrow \prod_{P \in X} W(F_P)$ on Witt groups has trivial kernel if and only if $\pi_1(\Gamma) = 1$. More generally, if $\pi_1(\Gamma)$ is free of rank r , then $\ker(\pi) = (\mathbb{Z}/2\mathbb{Z})^r$.

Corollary 9.6. *In the above situation, the following are equivalent:*

- (i) *The local-global principle for isotropy in Theorem 9.3 holds for **all** binary quadratic forms over F .*
- (ii) *The equality (*) of Corollary 9.4 holds for **all** quadratic forms over F .*
- (iii) *$F = F^{\text{split}}$ or equivalently, the reduction graph of a regular model of F is a tree.*

Proof. A form of dimension two is isotropic if and only if it is hyperbolic. So the equivalence of (i) and (iii) follows from Theorem 9.5, since the reduction graph Γ is a tree if and only if $\pi_1(\Gamma)$ is trivial.

By Corollary 9.4, the equality (*) is equivalent to the vanishing of the kernel of the local-global map on Witt groups. So the equivalence of (ii) and (iii) follows from Theorem 9.5 and Corollary 7.8. \square

Remark 9.7. As with Theorem 9.3, the other results above also have analogs in terms of patches on a regular model; the analog for Corollary 9.4 is in fact Corollary 4.3 of [HHK09].

We next turn to analogs of the above three results for the set of discrete valuations on F , thereby extending the results of [CPS08]. We first prove preliminary results, using the theorem of Springer cited above. For this reason we assume that the residue field k of the discrete valuation ring has characteristic unequal to two, as in [CPS08].

Lemma 9.8. *Let R be a regular complete local domain of dimension two, whose residue field k has characteristic unequal to two. Let E be the fraction field of R ; let $\{x, y\}$ be a generating set for the maximal ideal of R ; and let E_y be the completion of E with respect to the y -adic valuation. Let $q = \sum_{i=1}^n a_i Z_i^2$ be a diagonal quadratic form over R such that each a_i has the form $x^{r_i} y^{s_i} u_i$ for some $r_i, s_i \geq 0$ and some unit $u_i \in R^\times$.*

- (a) *If q is isotropic over E_y then it is isotropic over E .*
- (b) *If q is hyperbolic over E_y then it is hyperbolic over E .*

Proof. (a) We follow the strategy used in the proof of [CPS08, Theorem 3.1]. After factoring out even powers of x and y , we may assume that the exponents of x and y in the coefficients a_i of q are each equal to 0 or 1. We may now write $q = q_1 \perp xq_2 \perp yq_3 \perp xyq_4$, where each q_i is a regular quadratic form over R . Since $q = (q_1 \perp xq_2) \perp y(q_3 \perp xq_4)$ is isotropic over the complete discretely valued field E_y , it follows from a theorem of Springer ([Lam05, Proposition VI.1.9(2)]) that either $q_1 \perp xq_2$ or $q_3 \perp xq_4$ is isotropic over the residue field of E_y , i.e. over the fraction field of the discrete valuation ring $\bar{R} := R/yR$. Applying Springer's theorem to that subform over \bar{R} yields that one of the forms q_i is isotropic over the residue field k of \bar{R} , and hence so is q . Thus there is a non-zero k -point on the affine quadric hypersurface $Q \subset \mathbb{A}_R^n$ defined by q . By the assumptions on q and on k , this point of Q is smooth because it is not the origin. Since k is also the residue field of the complete ring R , [HHK09, Lemma 4.5] implies that this k -point lifts to an R -point of Q . Hence q is isotropic over E .

(b) By Witt decomposition, we may write $q = q' \perp q''$, where q' is anisotropic over E and q'' is hyperbolic over E . Since q and q'' are hyperbolic over E_y , so is q' , by Witt cancellation ([Lam05, Theorem I.4.2]). If q' is not the zero form, then it is isotropic over E_y , and hence it is isotropic over E by part (a). This contradiction shows that $q' = 0$ and hence $q = q''$ is hyperbolic. \square

In this proposition we return to our standing notation of this section.

Proposition 9.9. *Assume that the residue field k of T has characteristic unequal to two. Let F be a one-variable function field over the fraction field of T . Then for every quadratic*

form q over F , there is a regular model \widehat{X} of F such that for every point P on the closed fiber of \widehat{X} , q is isotropic (resp. hyperbolic) over F_P if and only if it is isotropic (resp. hyperbolic) over $(F_P)_v$ for every discrete valuation v on F_P .

Proof. The forward implication is true for any regular model, so it suffices to find a model for which the reverse implication holds. Let \widehat{X}' be any regular model for F . Since $\text{char}(F) \neq 2$, after replacing q by an equivalent form we may assume that q is a diagonal form $\sum a_i Z_i^2$, with $a_i \in F^\times$. Let D' be the union of the supports of the divisors of the elements a_i as rational functions on \widehat{X}' . For some blow-up $f : \widehat{X} \rightarrow \widehat{X}'$, the singularities of $D := f^{-1}(D')$ are normal crossings (e.g. see [HHK09, Lemma 4.7]). Let X be the closed fiber of \widehat{X} , and let $P \in X$. If P is the generic point of a component of X , then F_P is a complete discretely valued field, and the statement is trivial. So let P be a closed point. By the above condition on the divisor D , we may choose local parameters $x, y \in \widehat{R}_P$ such that the components of D that meet P lie in the zero locus of xy on \widehat{X} . After multiplying q by an element of the form $x^r y^s$, we may assume that q is as in the hypothesis of Lemma 9.8. If q is isotropic (resp. hyperbolic) over each completion of F_P , it is in particular isotropic (resp. hyperbolic) over the completion of F_P with respect to the y -adic valuation on F_P . Then the lemma implies that q is isotropic (resp. hyperbolic) over F_P , as we needed to show. \square

We now can now state analogs of Corollary 9.4, Theorem 9.5, and Corollary 9.6, in the context of discrete valuations. (As noted above, the corresponding analog of Theorem 9.3 was proven in [CPS08, Theorem 3.1].)

Theorem 9.10. *Assume that the residue field k of the complete discrete valuation ring T has characteristic unequal to two. Let F be a one-variable function field over the field of fractions K of T , and let Γ be the reduction graph of a regular model.*

- (a) *The quadratic forms on F satisfy a local-global principle for isotropy with respect to the set of discrete valuations Ω_F (i.e. an arbitrary form q is isotropic over F if it is isotropic over F_v for each $v \in \Omega_F$) if and only if Γ is a tree.*
- (b) *For any regular quadratic form q over F , its Witt index $i_W(q)$ is equal to either $\min_{v \in \Omega_F} i_W(q_v)$ or $\min_{v \in \Omega_F} i_W(q_v) - 1$. The second case occurs precisely for those forms that are Witt equivalent to an anisotropic binary form that becomes isotropic over each F_v . Moreover the first case holds for **all** regular quadratic forms q over F if and only if Γ is a tree.*
- (c) *The kernel of the natural homomorphism of Witt groups $\varpi : W(F) \rightarrow \prod_{v \in \Omega_F} W(F_v)$ is equal to the kernel of $\pi : W(F) \rightarrow \prod_{P \in X} W(F_P)$ for any regular model of F . Thus it is isomorphic to the elementary abelian two-group $\text{Hom}(\pi_1(\Gamma), \mathbb{Z}/2\mathbb{Z})$, and each element in the kernel is represented by a quadratic form of dimension two.*

Proof. (a) If the quadratic forms on F satisfy a local-global principle with respect to discrete valuations, then they also satisfy such a principle with respect to the points P on the closed

fiber of any regular model, since every F_v contains an F_P by Corollary 7.4. Hence Γ is a tree, by Corollary 9.6.

Conversely, suppose that Γ is a tree. Consider a quadratic form q over F that becomes isotropic over each F_v . We need to show it is isotropic over F . Let \widehat{X} be as in the conclusion of Proposition 9.9. Let P be a point on the closed fiber of \widehat{X} . Since every discrete valuation on F_P restricts to a discrete valuation on F (Proposition 7.5), the quadratic form q is isotropic over all completions of F_P with respect to its discrete valuations. Hence by Proposition 9.9, q is isotropic over each F_P . But then q is isotropic over F , by Corollary 9.6.

(b) Given a regular quadratic form q over F , let \widehat{X} be as in Proposition 9.9, with closed fiber X . By Corollary 9.4, it suffices to show that $\min_{v \in \Omega} i_W(q_v) = \min_{P \in X} i_W(q_{F_P})$. The former expression is greater than or equal to the latter, since every F_v contains an F_P ; and by part (a) they are equal if q has dimension two. It remains to show that the above inequality is actually an equality if $\dim(q) > 2$. In this case the right hand side is equal to $i_W(q)$, by Corollary 9.4. Write $q = q_a \perp q_h$ with q_a anisotropic and q_h hyperbolic. Since $i_W(q) = \min_{P \in X} i_W(q_P)$, the form q_a is anisotropic over F_P for some point $P \in X$. Given P , the form q_a remains anisotropic over the completion of F_P with respect to some discrete valuation, by Proposition 9.9. By Proposition 7.5 it also remains anisotropic over F_v for the restriction of that valuation to F . Hence $i_W(q_a) = i_W((q_a)_{F_P}) = i_W((q_a)_{F_v}) = 0$ and thus $i_W(q) = i_W(q_{F_v})$, as needed.

(c) Given any regular model \widehat{X} of F with closed fiber X , since every F_v contains an F_P by Corollary 7.4, it follows that $\ker(\pi)$ is contained in $\ker(\varpi)$.

Conversely, let \widehat{X} be as in Proposition 9.9, and consider a regular quadratic form q whose class lies in $\ker(\varpi)$. Again using Proposition 7.5, the class of q also becomes trivial over each completion of a field F_P (with respect to some discrete valuation), for each P in the closed fiber X of \widehat{X} . Proposition 9.9 then implies that the class of q lies in $\ker(\pi)$; implying $\ker(\varpi) \subseteq \ker(\pi)$ and hence equality. Now for an arbitrary regular model \widehat{X}' , the reduction graphs of \widehat{X} and \widehat{X}' have the same fundamental group by Corollary 7.8. Hence by Theorem 9.5, $\ker(\pi)$ and the corresponding kernel $\ker(\pi')$ for \widehat{X}' are isomorphic and finite. But since $\ker(\pi') \subseteq \ker(\varpi) = \ker(\pi)$, this implies $\ker(\pi') = \ker(\varpi)$, as desired. \square

9.3 Applications to central simple algebras

We next consider applications of our results to central simple algebras. We first carry over a local-global assertion about the index of an algebra from the context of patches to the contexts of points on the closed fiber and of valuations. Afterwards we prove a local-global result about central simple algebras with unitary involutions.

Recall that in [HHK09, Theorem 5.1] we showed that under Notation 3.2, the index of a central simple F -algebra A is the least common multiple of the indices of the central simple F_ξ -algebras $A_\xi := A \otimes_F F_\xi$, for $\xi \in \mathcal{P} \cup \mathcal{U}$. Using this, we obtain an analogous local-global principle for points on the closed fiber:

Theorem 9.11. *Under Notation 3.2, let A be a central simple F -algebra. Then*

$$\text{ind}(A) = \text{lcm}_{P \in X} \text{ind}(A_P), \quad (*)$$

where $A_P := A \otimes_F F_P$ is viewed as a central simple F_P -algebra, for $P \in X$. In particular, A is split over F if and only if each A_P is split over F_P .

Proof. Recall that if E/F is a field extension, then $\mathrm{SB}_i(A)(E) \neq \emptyset$ if and only if $\mathrm{ind}(A_E)$ divides i ([KMRT98], Proposition 1.17); here $\mathrm{SB}_i(A)$ is the generalized Severi-Brauer variety associated to A . Moreover, the rational connected linear algebraic group $\mathrm{GL}_1(A)$ acts transitively on the points of $\mathrm{SB}_i(A)$. So Theorem 9.1 implies that $\mathrm{ind}(A) \mid i$ if and only if $\mathrm{ind}(A_P) \mid i$ for each $P \in X$; this is equivalent to the desired assertion. \square

It would be desirable to prove an analog of the above theorem in terms of discrete valuations on F , parallel to the result in the case of quadratic forms ([CPS08, Theorem 3.1]). But that result relied on a theorem of Springer on quadratic forms ([Lam05, Proposition VI.1.9(2)]) for which there is no known parallel for central simple algebras. What can still be shown is a valuation version of the last assertion of the theorem, which can be viewed as an analog of the Albert-Brauer-Hasse-Noether theorem:

Corollary 9.12. *Let F be a one-variable function field over a complete discretely valued field, and let A be a central simple F -algebra. Then A splits over F if and only if it splits over F_v for every discrete valuation v on F . Moreover two such algebras A, A' are isomorphic over F if and only if they become isomorphic over each F_v .*

Proof. Let \hat{X} be a regular model for F over the valuation ring T of the complete discretely valued field K . The isomorphism classes of central simple F -algebras of degree d are classified by $H^1(F, \mathrm{PGL}_d)$ ([KMRT98], p. 396), with the isomorphism class of a split algebra corresponding to the trivial cohomology class. Since the connected linear algebraic group PGL_d is defined and reductive over \hat{X} , Theorem 8.10(ii) implies that $\mathrm{III}(F, \mathrm{PGL}_d) = \mathrm{III}_0(\hat{X}, \mathrm{PGL}_d)$. But $\mathrm{III}_0(\hat{X}, \mathrm{PGL}_d)$ is trivial by Theorem 9.11. Hence so is $\mathrm{III}(F, \mathrm{PGL}_d)$, and this implies the first assertion. The second assertion now follows from the first by considering the difference of the two Brauer classes, and using that A, A' have the same degree. \square

Remark 9.13. We sketch another proof of the above corollary, using ideas of [Sal97], Section 3 (see also [Sal98]). Namely, letting X_1, \dots, X_n be the irreducible components of the closed fiber X of a regular model \hat{X} of F , with generic points η_i and residue fields κ_i , the composition $\mathrm{Br}(\hat{X}) = \mathrm{Br}(X) \hookrightarrow \prod_i \mathrm{Br}(X_i) \hookrightarrow \prod_i \mathrm{Br}(\kappa_i)$ also factors as $\mathrm{Br}(\hat{X}) \rightarrow \prod \mathrm{Br}(\hat{R}_{\eta_i}) \rightarrow \prod_i \mathrm{Br}(\kappa_i)$. But any element $\alpha \in \mathrm{Br}(F)$ that splits over each F_v must lie in $\mathrm{Br}(\hat{X})$; and its image in $\prod_i \mathrm{Br}(\kappa_i)$ is trivial because it splits over each \hat{R}_{η_i} (using that we have an inclusion $\mathrm{Br}(\hat{R}_{\eta_i}) \hookrightarrow \mathrm{Br}(F_{\eta_i}) = \mathrm{Br}(F_{v_{\eta_i}})$). Since the first composition above is an inclusion, α is trivial, as asserted. The proof of the corollary above instead relies on [Nis84] (by using Theorem 8.10(ii) and hence Lemma 8.8), which can be viewed as generalizing this approach to more general reductive groups.

Turning to the second application, we consider central simple F -algebras with involutions, where F is as before.

Theorem 9.14. *Under Notation 3.2, let E/F be a quadratic étale algebra such that $E \otimes_F F_\wp \cong F_\wp \times F_\wp$ for every $\wp \in \mathcal{B}$. Let $(B, \tau), (B', \tau')$ be two central simple algebras with unitary involutions and centers isomorphic to E . Then (B, τ) and (B', τ') are isomorphic as algebras with involutions provided that they are locally isomorphic, in either of the following senses:*

- (a) *They become isomorphic over each field F_P for $P \in \mathcal{P}$, and for each field F_U for $U \in \mathcal{U}$.*
- (b) *They become isomorphic over each field F_P , for P a point on the closed fiber of \widehat{X} .*

Proof. (a) Let $\text{Aut}(B, \tau)$ denote the automorphism group of (B, τ) over F , and let $\text{Aut}_E(B, \tau)$ denote the automorphism group over E . Then $\text{Aut}_E(B, \tau)$ can be identified with $\text{PGU}(B, \tau)$; algebras with unitary involution and the same degree as B are classified by the cohomology set $H^1(F, \text{Aut}(B, \tau))$; and there is an exact sequence:

$$1 \rightarrow \text{PGU}(B, \tau) \rightarrow \text{Aut}(B, \tau) \rightarrow S_2 \rightarrow 1.$$

(See [KMRT98, page 400].) Here the map on the right is obtained by restricting the automorphism to E , and the symmetric group S_2 is identified with the group of automorphisms of E/F . In particular, the trivial element of $H^1(F, S_2)$ corresponds to the isomorphism class of E .

Suppose that (B, τ) and (B', τ') are as above and become isomorphic over each F_ξ for $\xi \in \mathcal{P} \cup \mathcal{U}$. Hence (B', τ') induces the trivial class in $H^1(F_\xi, \text{Aut}(B, \tau))$ for each $\xi \in \mathcal{P} \cup \mathcal{U}$, and hence it lies in $\text{III}_{\mathcal{P}}(\widehat{X}, \text{Aut}(B, \tau))$. Also, its image in $H^1(F, S_2)$ is trivial. So by Corollaries 2.6 and 3.4, this class lies in the image of $\text{III}_{\mathcal{P}}(\widehat{X}, \text{PGU}(B, \tau))$. To prove the assertion it thus suffices to show that $\text{III}_{\mathcal{P}}(\widehat{X}, \text{PGU}(B, \tau))$ is trivial.

To do this, we consider the exact sequence

$$1 \rightarrow R_{E_\wp/F_\wp}^1 \mathbb{G}_m \rightarrow \text{U}(B, \tau)_{F_\wp} \rightarrow \text{PGU}(B, \tau)_{F_\wp} \rightarrow 1.$$

Here $\text{PGU}(B, \tau)$ is the image of $\text{U}(B, \tau)$ in $\text{PGL}_1(B)/R_{E/F} \mathbb{G}_m$ ([KMRT98, p. 346]), where $R_{E/F}^1 \mathbb{G}_m$ is the torus of elements of E of norm 1. Since $E_\wp = F_\wp \times F_\wp$, we have $R_{E_\wp/F_\wp}^1 \mathbb{G}_m \cong \mathbb{G}_m$, via $(a, a^{-1}) \mapsto a$. Hilbert's Theorem 90 together with the six-term cohomology sequence associated to the exact sequence of groups above then tells us that the map $\text{U}(B, \tau) \rightarrow \text{PGU}(B, \tau)$ is surjective on F_\wp -points. So the map $\text{III}_{\mathcal{P}}(\widehat{X}, \text{U}(B, \tau)) \rightarrow \text{III}_{\mathcal{P}}(\widehat{X}, \text{PGU}(B, \tau))$ is surjective by Corollaries 2.6 and 3.4. But the rational group $\text{U}(B, \tau)$ is connected ([KMRT98, IV.23.A]); and so $\text{III}_{\mathcal{P}}(\widehat{X}, \text{U}(B, \tau))$ is trivial by Corollary 6.5 (this also follows from Theorem 3.7 of [HHK09]). Hence so is $\text{III}_{\mathcal{P}}(\widehat{X}, \text{PGU}(B, \tau))$.

(b) The class of (B', τ') defines an element of $H^1(F, \text{Aut}(B, \tau))$ that becomes trivial over each F_P . Thus the corresponding $\text{Aut}(B, \tau)$ -torsor has an F_P -point for each $P \in \mathcal{P}$. In particular, taking P to be the generic point η_i of an irreducible component X_i of X , Proposition 5.8 implies that there is an affine dense open subset U_i of X_i that does not meet any other component of X and such that the torsor has an F_{U_i} -point. Let \mathcal{U} be the collection of these sets U_i , and let \mathcal{P} be the complement of their union. Then the torsor has an F_ξ -point for every $\xi \in \mathcal{P} \cup \mathcal{U}$; and hence (B', τ') becomes isomorphic to (B, τ) over each F_ξ . The conclusion now follows from part (a). \square

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